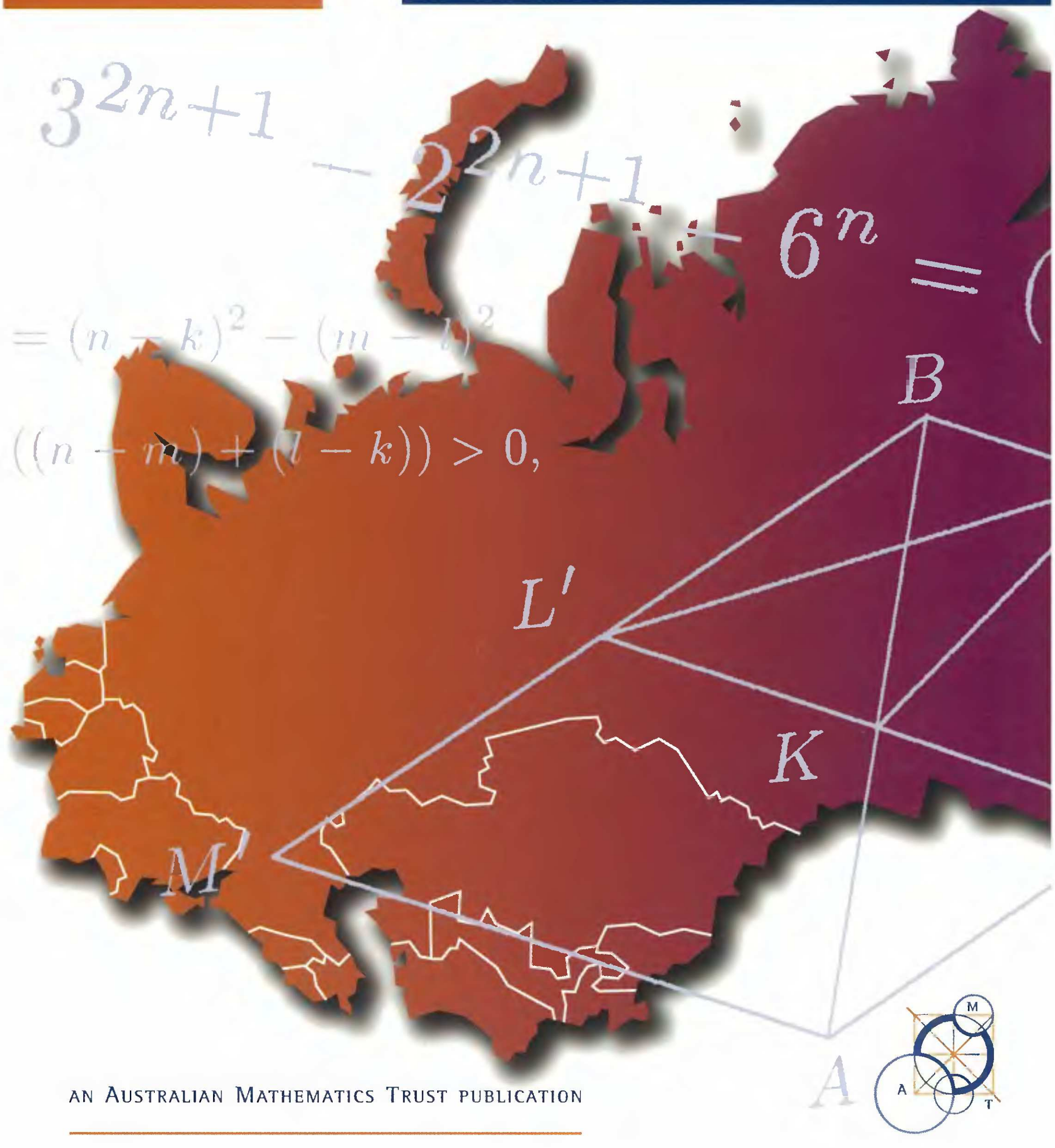
USSR MATHEMATICAL OLYMPIADS 1989-1992

AM SLINKO



AN AUSTRALIAN MATHEMATICS TRUST PUBLICATION

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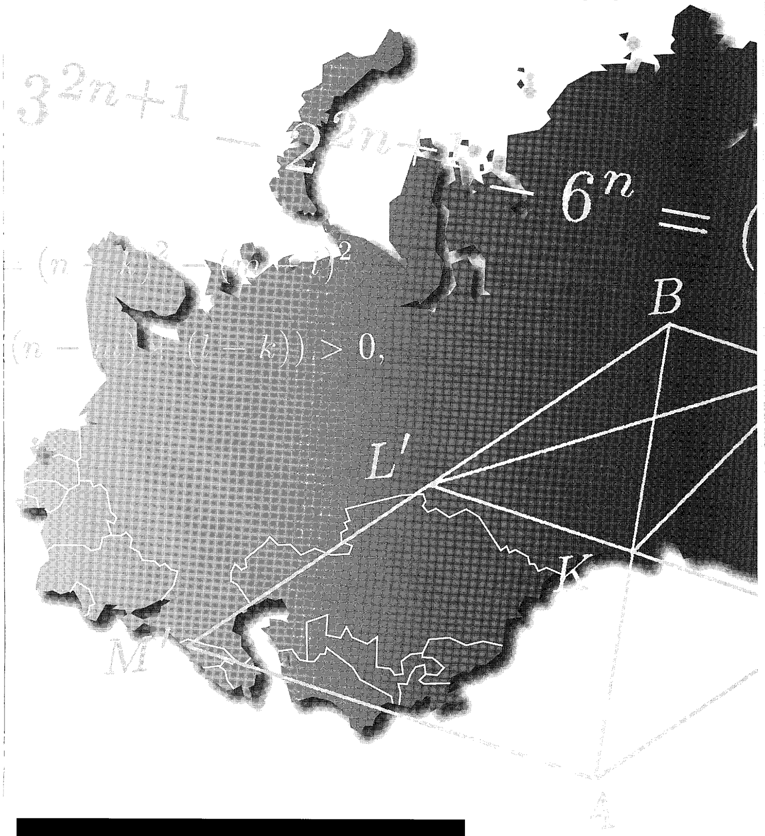

AM SLINKO

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FOREWORD

It has been claimed that Russian Mathematics flourished under Communism because Mathematics was less susceptible to political interference than any other intellectual pursuit and it consequently attracted many talented minds to its study. Whatever the reason, it must be conceded that Russia has produced more than its fair share of top quality mathematicians. What makes Russia unusual compared with just about any Western country is the large proportion of its top quality mathematicians who have devoted time and energy to the younger generation of rising mathematicians, via their involvement in problem-solving competitions and the associated training. Amongst these academicians there is an awareness of the necessity to challenge the cleverest students with difficult problems and to teach them the mathematical skills required for their solution, in order that they should maintain their enthusiasm for Mathematics and never be bored by it. It is arguable that the USSR Olympiad was the epitome of all national mathematical competitions. Its success was almost certainly due to the involvement of many mathematical experts, who somehow created remarkably clever problems aimed at particular grades of students. One perhaps should expect to be able to construct suitable problems for older grades who are equipped with some higher mathematics but I have always found it amazing that this competition is a rich source of challenging and interesting problems for younger grades. The variety of the problems, the suitability to age groups and the nature of the challenge has been of consistent excellent standard since the very first competition in 1967.

Arkadii Slinko is typical of the generation of mathematicians who generously pass on their enthusiasm for mathematics to the younger generation. He was, for many years, involved in the Russian Mathematical Olympiad Programme and has played a very important role in the New Zealand Mathematical Olympiad Programme over the last three years. At the 33rd IMO held in Moscow in 1992 Arkadii Slinko's Olympiad expertise was much appreciated by the international participants. As Chief Coordinator he assembled a magnificent team of marking coordinators which maintained a consistent high standard of assessment throughout the competition, and yet was always able to credit fruitful mathematical ideas without necessarily allocating marks. For the first time in the history of Olympiad Mathematics a certain Antipodean country achieved three grades of zero for the one problem. The second and third presented solutions, each having made slightly more progress than its predecessor, were awarded zero with the assurance that each zero was a better zero than its predecessor! I can still see Arkadii Slinko comforting the Team

Leader; at the same time fully supporting the decision of the coordinators. Arkadii Slinko has an amazing memory. I recall that at the 35th IMO in Hong Kong several Team Leaders were concerned that one proposed problem seemed familiar but they were unable to identify its original source. Arkadii Slinko arrived on the scene and immediately named the competition and year of its occurrence and furthermore, from memory alone, was able to name derived problems occurring in different competitions in following years.

Now, the whole of the international Olympiad community can benefit from Arkadii Slinko's expertise and his vast and intimate knowledge of the problems and solutions in this collection from the final four years of USSR Mathematical Olympiads. It will certainly be an invaluable volume to all those involved in the coaching of the World's talented young mathematicians.

David Paget
University of Tasmania
Australia, 1996

PREFACE

Mathematical competitions have a long history. According to the information available to us, as far back as ancient India, there were competitions in solving mathematical problems in the presence of numerous spectators. Mathematical tournaments were widely spread in the Middle Ages and later on in Italy and other European countries. As is widely known, in 1535 in Bologna, Niccolo Tartaglia won a mathematical competition in solving equations of the third degree. In 1594, Francois Viete won a contest, having solved one special equation of the 45th degree that was proposed as a challenge by Adrian van Roumen to all mathematicians. There are numerous other historical records of mathematical competitions, undertaken in person and by correspondence.

But notwithstanding their antiquity, such competitions are far from declining. While sportsmen have come close to the limits of physical ability, the abilities of human brains are inexhaustible.

The mathematical olympiads in secondary school started with the so-called "Eötvös's Competition" carried out in Hungary in 1894 at the initiative of Loran Eötvös, who was the president of the Hungarian Physical and Mathematical Society. In the USSR the first mathematical olympiad took place in Georgia. In 1933 schoolteachers Vashakidze and Petrakovskaja organized the first school and regional olympiads. The first town olympiads were conducted in 1934 at Tbilisi and Leningrad[1] and in the next year at Moscow and Kiev. In subsequent years olympiads spread all over the country. The noted mathematicians Delone and Tartakovsky (Leningrad), Shnirelman and Lyusternik (Moscow) and Kravchuk (Kiev) took part in organizing local olympiads and creating problems.

On the national scale, the idea of organizing a mathematical olympiad matured by 1960. Two renowned mathematicians, Kolmogorov (Moscow) and Lavrent'ev (Novosibirsk), played parts of great importance in this consolidating process. Kolmogorov organized, in 1961, the All-Russian Mathematical Olympiad; and Lavrent'ev, in 1962, organized the All-Siberian Mathematical Olympiad. These competitions did not fully correspond to their names, since the first involved students of the European part of the country, including students of some Republics, while the second involved students of Siberia, Far East and some Central Asiatic Republics. Although the delegations of the Republics were usually called teams, all competitions were individual. Each Republic had its own quota, which had been in proportion with the total number of its

[1]Currently and before 1914 known as St.Petersburg.

citizens. Moscow with its 9 million of population enjoyed the rights of a small republic.

Kolmogorov and Lavrent'ev also founded two special high schools in Moscow (1961) and Novosibirsk (1962). They were set up to teach mathematically gifted students who showed their excellence during the olympiads. Professors of Moscow and Novosibirsk State Universities elaborated curricula for those schools, and delivered lectures. At that time all schools in the Soviet Union were standard and they taught in conformity with the curriculum approved by the Ministry of Education. As a characteristic feature of that Ministry, it can be mentioned that a large number of wives of the Soviet Army and KGB top officers used to work there. The small exception had been made for language-special schools, where students had additional lessons in foreign language and where Soviet officials of different standing used to educate their offspring. Referring to this precedent, Kolmogorov was able to get permission to found a mathematical-special high school, making a major breach in the concrete wall of the Ministry of Education. Within a short time a whole network of special high schools (not only mathematical) that roughly corresponded to the network of olympiads was established.

A few words may be necessary in order to explain how Academician Lavrent'ev and his high school appeared in Siberia, where western people used to picture only bears and prisoners' camps. This picture is far from being true. During World War 2 many industrial enterprises were evacuated to behind the Urals. As a consequence, in the fifties Siberian towns such as Novosibirsk, Krasnoyarsk and Chelyabinsk were highly-populated industrial giants.

The Communist Party leader of that time, Nikita Khrushchev, was very active beyond the limits of his competence. He taught artists how to paint pictures and biologists how to raise the best strains of cattle, and supported the notorious academician Lysenko, the Father of Marxist-Leninist biology. The intellectual atmosphere was not favourable in Moscow. That led three academicians, Khristianovich, Lavrent'ev and Sobolev, to the idea of organizing a large-scale scientific centre outside Moscow and far from it. A picturesque place on the shore of the artificial Ob Sea near Novosibirsk was chosen. Of course, such an idea could not be converted into a reality without the permission of the same Khrushchev and the Politbureau, but Lavrent'ev managed to get them to agree. The Siberian branch of the Academy of Sciences came into being.

Since the intellectual atmosphere in the new Academy-town was very comfortable for scientists, many of them moved to Siberia from various parts of USSR. Those who were there at that time remember it with great nostalgia. I was then aged 15. I was an active participant of the Second All-Siberian Olympiad and was admitted to the Lavrent'ev High School, where I studied in 1963–1965.

Finally in 1967 the first All-Union Mathematical Olympiad took place. This was the capstone, which completed the pyramid of olympiads in the USSR. The integrating process was over. In its final stage the All-Union Mathematical Olympiad involved teams from 15 Republics, teams from Moscow and Leningrad, and teams from some outstanding special high schools. Students of Moscow and Leningrad, as well as the students of special high schools, were singled out in separate teams because they had the highest level of knowledge and sometimes special training. This separation permitted gifted students without special training to successfully climb the pyramid of olympiads to its highest vertex, and only then to meet in competition with the students of the special high schools.

Only students from the highest 3 forms, of ages 15 to 17, participated in the All-Union Olympiad. In 1989 they were from forms 8–10, and later, after the introduction in 1990 of one preparatory year, they were from forms 9–11. Traditionally it was a two-day competition, and each day students were asked to solve 4 problems in 4 hours. Although sometimes one problem could be given to students of 2 or more forms, normally 24 new problems needed to be composed every year. For this very difficult task the Methodological Committee, consisting of 20–30 professional mathematicians and educationists, was set up by the Ministry of Education. The members of this Committee each year formed a permanent half of the Jury of the All-Union Olympiad. The other half was formed by local mathematicians, educationists and teachers.

It became a tradition that the republics took turns in organizing the All-Union Olympiads. As a result the competition traveled from one Soviet republic to another, and usually it had an impact on the olympiad movement in these republics. I joined the Methodological Committee and the Jury in 1977, when the olympiad took place in Tallinn (Estonia), and was lucky to remain a member of it until the last one in 1992 which took place in Alma-Ata (Kazakhstan). During this period the only republics which I did not visit with the olympiad were Armenia and Lithuania.

Creating problems for the All-Union Olympiad was not the only responsibility of the Methodological Committee. Each year we also produced 20–25 new problems, which the Ministry of Education distributed between Republican Ministries of Education as guidelines for their republican olympiads. Different republics used them differently. Some heavily relied on them in their preparations of local olympiads. Some, like Latvia, composed their own problems and used the problems composed by the Methodological Committee only for the team selection to the All-Union Olympiad.

The 23rd All-Union Olympiad in Riga, which took place in 1989, was

especially well-organised and successful. As an experiment, in the framework of this olympiad but as a separate competition, a Computer Tournament was set up where students had to solve mathematical problems with the use of a computer. The experiment was a success, but it lasted only one more year and died due to the lack of resources. We didn't know then in Riga that it was the heyday of the All-Union Olympiad. Four years later, in 1992, the USSR disintegrated. However, due to inertia, since money was allocated already, the Olympiad in 1992 took place under the name of the 1st Mathematical Olympiad of the Commonwealth of Independent States (CIS). The second has never taken place.

In 1992, in the chaotic environment of a changing Russia, the 33rd International Mathematical Olympiad took place in Moscow. I was appointed as the Coordinator-in-Chief of this Olympiad. At this stage I had already signed a contract with the University of Auckland and for me it was a farewell to Russia. The team of 45 coordinators did a very good job which has been acknowledged on a number of occasions since. The problems of IMO-92 the reader will find at the end of the book.

In the history of Russia, and maybe in the history of olympiads in Russia, a new page has opened. A group of enthusiastic mathematicians in difficult economic conditions are struggling to maintain the standards which were set by the All-Union Olympiads. Let us wish them good luck.

A.M.S.
Auckland
New Zealand
December 1996.

ACKNOWLEDGEMENTS

Since every author writes about his own experience and events in which he personally participated, it is always easy, voluntarily or not, to make an impression that the author's role was much greater than it was in reality. Having in mind that up to a 100 mathematicians took part in organizing USSR Mathematics Olympiads at different stages of their existence, the reader should be absolutely convinced that the authors role was a somewhat humble one.

I first joined the Jury in 1977. Under the supervision of A Kolmogorov the leading figures of the Jury at that time were: M Bashmakov, I Bernshtein, A Egorov, S Fomin, V Gutenmaher, Yu Ionin, B Ivlev, N Konstantinov, L Makar-Limanov, A Plotkin, A Savin, M Serov, N Tolpigo, N Vasil'ev, A Zemlyakov and others. In the beginning of the eighties the Jury had changed significantly. Some had retired from math olympiads and concentrated on their research, some had immigrated, and some, like N Konstantinov, after disagreement with the Ministry of Education stepped down from the Jury and organized an alternative competition which is now known as the Tournament of Towns. The leadership was passed on to N Rozov, Yu Nesterenko, V Vavilov, S Režnichenko, I Sergeev, L Kuptzov, A Fomin, S Konyagin and myself. The charming B Gnedenko served as the Chairman of the Jury after Kolmogorov. Among other key members were N Agahanov, A Andzans, V Bernik, A Berzinsh, B.Chinik, S Duzhin, D Flaas, S Gashkov, V Golubyatnikov, A Grishin, P Gusyatnikov, E Khukhro, A Kolotov, O Lyashko, A Merkur'ev, D Mit'kin, O Musin, N Netzvetaev, S Pchelintzev, V Prasolov, I Shestakov, I Sharygin, Yu Solov'ev, D Tereshin, V Uroev, I Voronovich, I Zhuk. Many of them were prolific authors of original questions for math olympiads. Most of these mathematicians are my personal friends and I am greatly indebted to all of them.

My sincere thanks to Garry Tee for his invaluable help, his editor's remarks significantly improved the manuscript, to Patricia Fauring for reading and commenting on the preliminary version of the manuscript and to Prof Peter Taylor and Dr Andrei Storozhev for their encouragement and help with the pictures.

A.M.S.
January, 1997

From the Publisher

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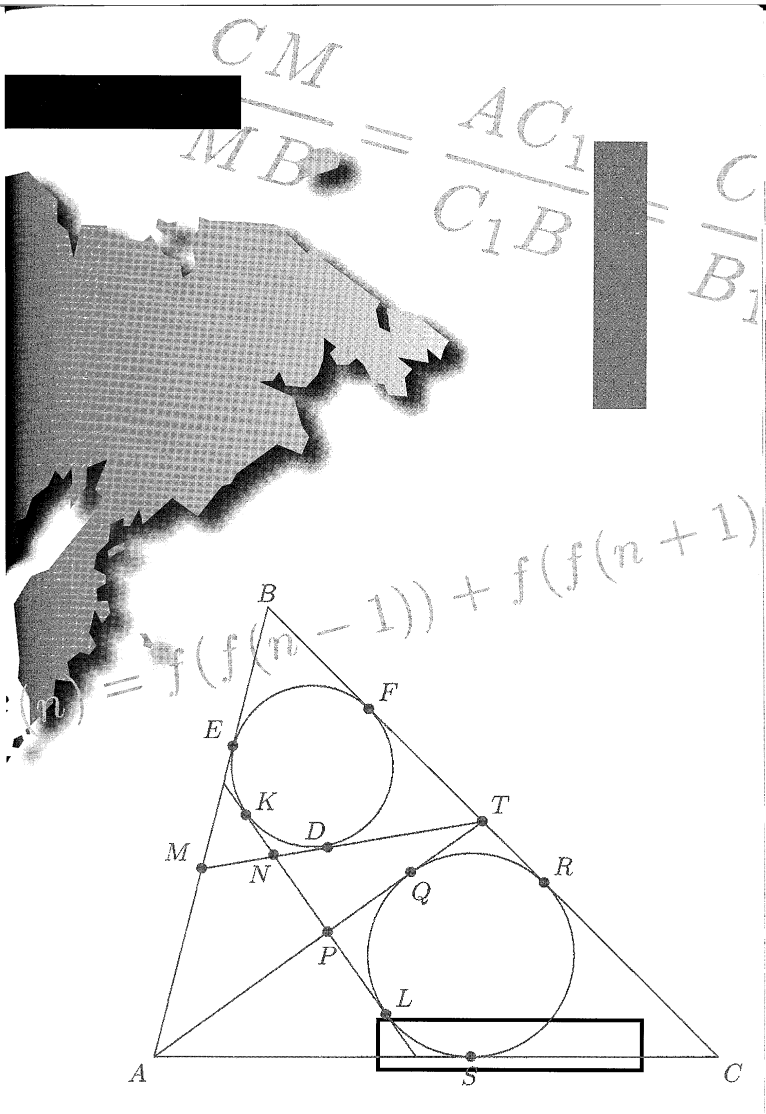
P.J.T.
April 1997



The author addresses the participants of the 1989 USSR Olympiad in Riga at the Opening Ceremony.

USSR OLYMPIAD 1989

8 FORM

First day

1. On Sunday each of 7 boys went to the ice-cream shop 3 times. It is known that each pair of them met at the shop. Prove that at a certain time 3 of them met there.

(A Andjans, Riga)

2. There are 77 right-angled blocks of dimensions $3 \times 3 \times 1$. Is it possible to place all these blocks in a closed rectangular box of dimensions $7 \times 9 \times 11$?

(A Berzinsh, Riga)

3. Let M be the point of tangency of the incircle of triangle ABC with side AB. Let T be an arbitrary point on side BC different from the vertices. Prove that the three circles inscribed in triangles BMT, MTA, ATC are tangent to a common line.

(A Andjans, Riga)

4. A natural number N has exactly 12 divisors (including 1 and N), which are numbered in increasing order $d_1 < d_2 < \ldots < d_{12}$. It is known that the divisor with index $d_4 - 1$ is equal to the product $(d_1 + d_2 + d_4)d_8$. Find N.

(A Berzinsh, Riga)

8 FORM

Second day

5. Eight pieces are placed on a chessboard so that each row and each column contains exactly one piece. Prove that there are an even number of pieces on the black squares of the board.

(V Proizvolov, Moscow)

6. On the sides AB, BC, CA of a triangle ABC points C_1, A_1, B_1, different from the vertices, are marked with the colour green. It turns out that

$$\frac{AC_1}{C_1B} = \frac{BA_1}{A_1C} = \frac{CB_1}{B_1A},$$

and $\angle BAC = \angle B_1A_1C_1$. Prove that the triangle with the green vertices is similar to the triangle ABC.

(I Sharygin, Moscow)

7. In a certain forest there are $n \geq 3$ starling-houses, and the distances between pairs of them are all different. In each starling-house there lives exactly one bird. At a certain time some of the birds fly to other starling-houses, so that in each starling-house there is again exactly one bird. This was done in such a way that if the initial distance between one pair of birds was less than that of another pair (the same bird can be considered in different pairs), then after the flight the distance between the first pair turned out to be greater than that of the second. For what values of n is this possible?

(A Berzinsh, Riga)

8. Consider the set of all five-digit numbers whose decimal representation is a permutation of the digits $1, 2, 3, 4, 5$. Prove that this set can be divided into two groups, in such a way that the sum of the squares of the numbers in each group is the same.

(D Fomin, St. Petersburg)

9 FORM

First day

9. We have 2000 coins. Two of them are false: one is lighter and the other is heavier than real ones. It is possible to perform no more than four weighings using a balance without weights. Show how to detemine which is greater, the sum of the weights of the two counterfeit coins or the sum of the weights of two real coins, or whether both of these quantities are the same.

(S Augustinovich, Novosibirsk)

10. Prove that if a, b, c are the lengths of the sides of a triangle and $a + b + c = 1$, then the following inequality holds

$$a^2 + b^2 + c^2 + 4abc < \frac{1}{2}.$$

(D Tereshin, Moscow)

11. Points K and M are chosen on the sides AB and CD respectively of a convex quadrilateral $ABCD$. Let L be the point of intersection of segments AM and KD, and let N be the point of intersection of segments KC and BM.

(a) Prove that if K and M are the midpoints of AB and CD, then

$$\text{Area}\, KLMN < \frac{1}{3}\text{Area}\, ABCD.$$

(b) Prove that if $AK : KB = CM : MD = m : n$, then

$$\text{Area}\, KLMN < \frac{mn}{m^2 + mn + n^2}\,\text{Area}\, ABCD.$$

(D Tereshin, Moscow)

12. A 23×23 square was formed from several squares of sizes 1×1, 2×2, 3×3. What is the least number of 1×1 squares needed to do this?

(N Agahanov, Moscow)

9 FORM

Second day

13. Do there exist real numbers a and b such that

(a) $a+b$ is rational and a^n+b^n is irrational for each natural $n \geq 2$;

(b) $a+b$ is irrational and a^n+b^n is rational for each natural $n \geq 2$?

(N Agahanov, Moscow)

14. A fly and a spider are on a 1 metre by 1 metre square ceiling. In one second the spider can jump from its position to the middle of any of the four segments which join it to the vertices of the ceiling. The fly does not move. Prove that in eight seconds the spider can be within 1 centimetre of the fly.

(V Il'ichev, Moscow)

15. The lateral sides AB and CD of a trapezium $ABCD$ are equal. The triangle $A'B'C$ is obtained from the triangle ABC by rotating it about the point C through a certain angle. Prove that the midpoints of the segments $A'D$, BC, $B'C$ lie on a straight line.

(V Protasov, Kiev)

16. An infinite sheet is divided into squares by two sets of parallel lines. Prove that for each natural number n there exists a polygon (not necessarily convex), whose sides go along the lines of the grid, which can be cut into 2 by 1 rectangles in exactly n different ways.

(B Kukushkin, Moscow; D Tulyakov, Moscow)

10 FORM

First day

17. What is the least natural number n for which equation

$$\left[\frac{10^n}{x}\right] = 1989$$

has an integer solution? (By $[a]$ we mean the integer part of a number a, that is, the largest integer which is not greater than a.)

(S Gashkov, Moscow)

18. Points D, E, F are chosen on the sides AB, BC, AC of a triangle ABC, so that $DE = BE$ and $FE = CE$. Prove that the centre of the circle circumscribed around triangle ADF lies on the bisector of angle DEF.

(V Protasov, Kiev)

19. Two points A and B are chosen on one of two intersecting spheres, and two points C and D are chosen on the other one. The segment AC passes through a common point of the spheres. The segment BD passes through another common point of the spheres, and is parallel to the line passing through the centres of the spheres. Prove that the projections of the segments AB and CD on the line AC are equal.

(Sharygin, Moscow)

20. Two tourists are at the same altitude, at points A and B which lie on opposite sides of a mountain chain. A path joining from point A to point B has the form of a broken line whose vertices are higher than the end-points A and B. Is it possible for the two tourists to travel along the whole path from one end to the other, so that at all times their altitudes are equal?

(E Abakumov, Moscow; D Fomin, St Petersburg)

10 FORM

Second day

21. Find the least value of the expression $(x+y)(y+z)$, given that x, y, z are positive numbers satisfying the equation

$$xyz(x+y+z) = 1.$$

(O Khristenko, Moscow)

22. A polyhedron with an even number of edges is given. Prove that an arrow can be placed on each edge so that each vertex is pointed at by an even number of arrows.

(O Lyashko, Moscow; O Musin, Moscow)

23. Does there exist a function $f(n)$, which maps the set of natural numbers into itself such that for each natural number $n > 1$ the following equation is satisfied

$$f(n) = f(f(n-1)) + f(f(n+1))?$$

(E Barabanov, Minsk; I Voronovich, Minsk)

24. A convex polygon is given. A median is any segment drawn between a vertex of the polygon and a point on its boundary, which divides the polygon into two parts of equal area. It is known that all medians of the polygon have length less than 1. Prove that the area of the polygon is less than $\pi/4$.

(S Anisov, Moscow; D Tulyakov, Moscow)

SOLUTIONS

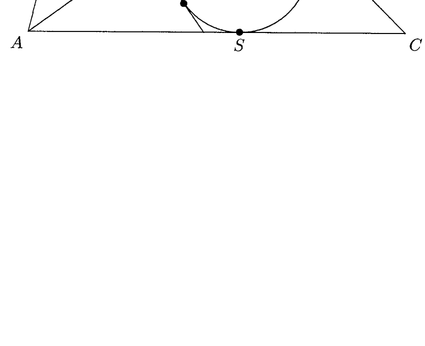
1. Without loss of generality we may assume that no two boys arrived at the shop at the same time. We can reduce the general case to this one by making some of the visits shorter.

We know that each of the 7 boys went to the shop 3 times. In all there were 21 visits. Let $t_1 < t_2 < \ldots < t_{21}$ be the times they arrived. If we suppose that there was no time when 3 of the boys were in the shop simultaneously, a boy approaching the shop at time t_i, $i \geq 2$, could see no more than one friend in the shop. The boy that came to the shop at time t_1 saw nobody. In all, no more than 20 pairs of boys met at the shop. But the total number of pairs is $\frac{7\times 6}{2} = 21$. This gives a contradiction.

2. Answer: no.

If it were possible then the blocks would fill up the box. Let us consider the layer of thickness 1 adjacent to a face of size 7×11. Each block either lies entirely within this layer, or it fills a region of size $3 \times 1 \times 1$. Therefore, the number of $1 \times 1 \times 1$ cubes in this layer must be divisible by 3, but the number $7 \times 11 = 77$ is not divisible by 3.

3. Let l be a tangent to the circles inscribed in the triangles BMT and ATC, and let N and P be the points where l intersects the segments MT and AT.



The problem will be solved if we can show that it is possible to inscribe a circle in the quadrilateral $AMNP$, and for this it is sufficient to prove that $MN+AP = AM+NP$. Since the distances from the intersection of the two tangents to a circle to the points of tangency are equal, the following equality holds

$$\begin{aligned} MN + AP - NP &= (MD - KN) + (AQ - PL) - NP \\ &= ME + AS - (KN + NP + PL) \\ &= ME + AS - KL. \end{aligned}$$

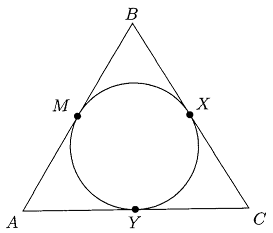
Since

$$KL = FR = BC - BE - CS,$$

we obtain

$$MN + AP - NP = AB + AC - BC - AM.$$

It remains only to observe that in the following diagram



$$\begin{aligned} AB + AC - BC &= (AM + AY) + (MB - BX) + (YC - CX) \\ &= AM + AY = 2AM. \end{aligned}$$

4. Answer: $N = 1989$.

The numbers $N/d_1, N/d_2, ..., N/d_{12}$ are all divisors of N written in decreasing order, thus $N/d_1 = d_{12}$, $N/d_2 = d_{11}$, ..., $N/d_{12} = d_1$. Therefore, for each positive integer $i \leq 12$ we have

$$d_i d_{13-i} = N.$$

Denote $d_4 - 1$ by k. By hypothesis we know that the number $d_1+d_2+d_4$ is a divisor of d_k. Therefore this number is a divisor of N with an index which is not less than 5, and hence $d_1+d_2+d_4 \geq d_5$. Consequently, since $d_5 d_8 = N$, we have

$$d_k = (d_1 + d_2 + d_4)d_8 \geq d_5 d_8 = N.$$

Since $d_k \leq N$ we obtain $d_k = N$. That is $k = 12$, $d_4 = 13$, and

$$d_5 = d_1 + d_2 + d_4 = d_2 + 14.$$

Since $d_2 \leq d_4 - 2 = 11$ and is obviously a prime number, we have the following possible cases:

(a) $d_2 = 2$. Then $d_5 = 16$, and consequently N has divisors $1, 2, 4, 8$ and thus $d_4 \leq 8$, which is false.

(b) $d_2 = 5$. Then $d_5 = 19$, which means that N has prime divisors 5, 13 and 19. N cannot have more than 3 prime divisors otherwise the total number of divisors of N would be greater than 12. Therefore, it has exactly 3 prime divisors, but in this case d_3 would be the product of some of these divisors and so would be greater than d_4, which is false.

(c) $d_2 = 7$. Then $d_5 = 21$ and N is divisible by 3, so that $d_2 \leq 3 < 7 = d_2$ which is a contradiction.

(d) The case $d_2 = 11$ can be dealt with in the same way as the previous case.

(e) $d_2 = 3$. Then $d_5 = 17$, and N has prime divisors $d_2 = 3$, $d_4 = 13$ and $d_5 = 17$. Since, as was noticed earlier, there can be no other prime divisors of N, $d_3 = 9 = 3^2$. Consequently, N is divisible by $9 \times 13 \times 17 = 1989$, and since this number has exactly 12 divisors then $N = 1989$. It is easy to check that $N = 1989$ satisfies all the conditions of the problem.

5. Place ones, twos and threes on the chess board as shown in the diagram.

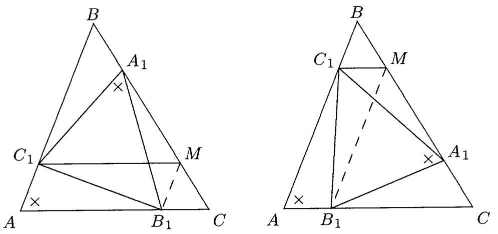
We refer to the black squares with odd numbers as squares of the first kind, the black squares with even numbers as squares of the second kind, and the white squares with odd numbers as squares of the third kind.

Suppose that there are n_i pieces on the squares of the ith kind.

	2		2		2		2
1	3	1	3	1	3	1	3
	2		2		2		2
1	3	1	3	1	3	1	3
	2		2		2		2
1	3	1	3	1	3	1	3
	2		2		2		2
1	3	1	3	1	3	1	3

From the conditions of the problem we have that $n_1 + n_3 = 4$ and $n_2 + n_3 = 4$. Therefore $n_1 = n_2$, and the number of pieces on black squares is equal to $n_1 + n_2 = 2n_1$ which is an even number.

6. Draw C_1M parallel to AC (the two possible cases are shown below).



By Thales' Theorem and the conditions of the problem we have

$$\frac{CM}{MB} = \frac{AC_1}{C_1B} = \frac{CB_1}{B_1A}.$$

It follows that B_1M is parallel to AB and, therefore AC_1MB_1 is a parallelogram. Consequently, $\angle C_1MB_1 = \angle CAB = \angle C_1A_1B_1$ and the points A_1, B_1, C_1, M lie on a circle. If M lies on the segment A_1C (see first diagram), then $\angle C_1B_1A_1 = \angle C_1MA_1 = \angle ACB$. If M lies on the segment A_1B (see second diagram), then $\angle C_1B_1A_1 = 180^\circ - \angle C_1MA_1 = \angle C_1MB = \angle ACB$. In either case the triangles ABC and $A_1B_1C_1$ have equal angles and therefore are similar.

7. Answer: only for $n = 3$.

We denote the starling-houses by $M_1, \dots, M_n$, the distance between M_i and M_j by $d(M_i, M_j)$, and by $f(M_i)$ the starling-house to which the bird from M_i flies by $f(M_i)$. Then f is a one-to-one correspondence between the set $\{M_1, \dots, M_n\}$ and itself. Moreover, if $d(M_i, M_j) < d(M_k, M_l)$, then $d(f(M_i), f(M_j)) > d(f(M_k), f(M_l))$; and therefore

$$d(f^2(M_i), f^2(M_j)) < d(f^2(M_k), f^2(M_l)),$$

where f^2 denotes the mapping f applied twice. It follows that if $d(f^2(M_p), f^2(M_q))$ is the smallest of all the distances of the form

$$d(f^2(M_i), f^2(M_j)), \quad i \neq j,$$

then $d(M_p, M_q)$ is the smallest of the distances $d(M_i, M_j)$, $i \neq j$. Since these 2 sets of values are the same, their smallest members coincide. Therefore,

$$d\left(f^2(M_p), f^2(M_q)\right) = d(M_p, M_q).$$

In a similar fashion (or by induction which is now trivial) it can be established that

$$d(f^2(M_i), f^2(M_j)) = d(M_i, M_j),$$

for all i and j. Since the set of distances from a given point to the rest of the points determines the given point uniquely, we conclude that f^2 is the identity mapping. Therefore, f is such that if $f(M_i) = M_j$, $i \neq j$, then $f(M_j) = M_i$. If we allow n to be greater than 4, then we can find 2 different pairs of points (M_i, M_j) and (M_k, M_l), such that in each pair either the points are fixed under f or are interchanged by f. In either case the distance between the two points in each pair remains fixed. This is not possible, since according to the hypothesis if

$$d(M_i, M_j) < d(M_k, M_l)$$

then

$$d(f(M_i), f(M_j)) > d(f(M_k), f(M_l))$$

while we have

$$d(f(M_i), f(M_j)) = d(M_i, M_j)$$

and

$$d(f(M_k), f(M_l)) = d(M_k, M_l).$$

Thus, the conditions of the problem can be fulfilled only for $n = 3$. Let us check that this is indeed possible. Let $a = d(M_1, M_2)$, $b = d(M_2, M_3)$ and $c = d(M_3, M_1)$, where $a < b < c$. Then if $f(M_1) = M_1$, $f(M_2) = M_3$ and $f(M_3) = M_2$, the conditions of the problem are fulfilled. Consequently, $n = 3$ is in fact a solution to the problem.

8. We say that two sets of k-digit numbers are friendly if each set contains the same number of elements and the sum of these elements as well as the sum of their squares is the same for each set. To solve the problem, it is sufficient to show that the set of all 5-digit numbers composed from the digits $1, 2, 3, 4, 5$, each used only once, can be divided into 2 friendly sets.

Firstly we shall note that the set of all 3-digit numbers composed from any 3 different digits a, b, c, none of which is zero, can be divided into 2 friendly sets. Indeed, each of the sets $\{\overline{abc}, \overline{cab}, \overline{bca}\}$, $\{\overline{acb}, \overline{bac}, \overline{cba}\}$ has 3 elements, the sum of the numbers in each set is equal to $111(a + b + c)$ and the sum of their squares is equal to $10101(a^2 + b^2 + c^2) + 2220(ab + ac + bc)$. We will need to apply the following argument twice. So we formulate it as a lemma.

Lemma. Let $\{A_1, \ldots, A_n\}$ and $\{B_1, \ldots, B_n\}$ be 2 friendly sets of k-digit numbers, composed from the digits $a_1, \ldots, a_k$ so that each digit is used only once. Let a be some other digit, different from zero and from $a_1, \ldots, a_k$. Then the two sets $\{U_1, \ldots, U_n\}$ and $\{V_1, \ldots, V_n\}$ of $(k + 1)$-digit numbers, where $U_i = A_i + a10^k$ and $V_i = B_i + a10^k$, $i = 1, 2, ..., n$, are also friendly.

Proof. It is clear that the sum of the U_i's and the sum of the V_i's are both equal to $S + an10^k$, where S is the common sum of all A_i's and B_i's. Also

$$\begin{aligned} U_1^2 + \ldots + U_n^2 &= \left(A_1 + a10^k\right)^2 + \ldots + \left(A_n + a10^k\right)^2 \\ &= \left(A_1^2 + \ldots + A_n^2\right) + 2a\left(A_1 + \ldots + A_n\right)10^k + na^2 10^{2k} \\ &= \left(B_1^2 + \ldots + B_n^2\right) + 2a\left(B_1 + \ldots + B_n\right)10^k + na^2 10^{2k} \\ &= V_1^2 + \ldots + V_n^2, \end{aligned}$$

and the lemma is proved.

Using the lemma for the case $k = 3$, we find that the set of all 4-digit numbers whose decimal representation consists of 4 different digits a, b, c, d, none of which is zero, with the digit d as the first digit (representing thousands), can be divided into 2 friendly sets. Analogously, 3 other similar sets, the sets of all 4-digit numbers whose decimal representation consists of a, b, c, d, with either the digit a, b or c as the first digit, can be divided into 2 friendly sets. Thus, we obtain 4 pairs of friendly sets (S_1, T_1), (S_2, T_2), (S_3, T_3), (S_4, T_4), such that the union of all eight sets in those pairs is the whole set of 4-digit numbers whose decimal representation consists of a, b, c, d. As a result we obtain a partition of all the numbers under consideration into 2 sets $S = S_1 \cup S_2 \cup S_3 \cup S_4$ and $T = T_1 \cup T_2 \cup T_3 \cup T_4$, which are obviously friendly.

Clearly the same argument works for $k = 4$ also. In particular, letting $a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4$ and $a = 5$ we get a solution to the problem.

9. Divide the coins into 4 groups, each containing 500 coins, and denote the total weights of the coins in each of these groups by A_1, A_2, A_3, A_4. Compare A_1 with A_2 and A_3 with A_4. It is easy to see that all possible cases reduce to the following three:

(a) $A_1 = A_2$, $A_3 = A_4$. Then the 2 false coins lie in the same group, and the sum of their weights is equal to the sum of 2 real coins. In this case two weighings are sufficient.

(b) $A_1 = A_2$, $A_3 > A_4$. In this case we form 2 new groups, uniting the first group with the second one and the third with the fourth. Comparing the weight of these two groups, we easily arrive at an answer. In this case, therefore, 3 weighings are sufficient.

(c) $A_1 > A_2$, $A_3 > A_4$. In this situation there are two possibilities. Either the heavy coin lies in the first group and the light one in the fourth group, or the heavy coin lies in the third group and the light one in the second. Comparing the weights of A_1 and A_4, we know which of these two subcases we are in. Afterwards, we compare $A_1 + A_4$ with $A_2 + A_3$ and we obtain the final answer. In this case we need 4 weighings.

10. Since the semi-perimeter of the triangle is $1/2$, its area, according

to Heron's formula, is

$$S = \sqrt{\frac{1}{2}\left(\frac{1}{2}-a\right)\left(\frac{1}{2}-b\right)\left(\frac{1}{2}-c\right)}.$$

From this and the fact that $a+b+c=1$ it follows that

$$16S^2 = -1 + 4(ab+bc+ca) - 8abc.$$

Squaring both sides of the equation $a+b+c=1$, we get that

$$2(ab+bc+ca) = 1-(a^2+b^2+c^2),$$

and therefore,

$$\begin{aligned} 16S^2 &= 1-2\left(a^2+b^2+c^2\right)-8abc \\ &= 2\left(\frac{1}{2}-a^2-b^2-c^2-4abc\right) > 0. \end{aligned}$$

From which we obtain the necessary inequality.

Remark. If a,b,c are the sides of a triangle and p,r,R are the semiperimeter, the radius of the inscribed circle and the radius of the circumscribed circle respectively, then it is known that a,b,c are the roots of equation

$$x^3 - 2px^2 + (p^2+r^2+4rR)x - 4prR = 0.$$

(You can prove this using trigonometry.) Hence the following equations hold:

$$a^2+b^2+c^2 = 2(p^2-r^2-4Rr), \quad abc = 4pRr.$$

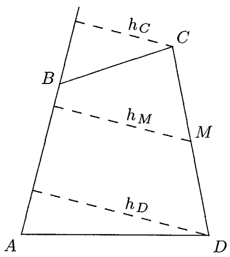
Since $p=1/2$, it follows that

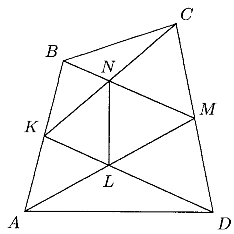
$$a^2+b^2+c^2+4abc = \frac{1}{2}-2r^2.$$

Therefore, we see that the given inequality is exact.

11. (a) The statement in section (a) is the case $m=n=1$ of that in section (b). Nevertheless, we present a direct geometrical proof of this inequality. Let h_M, h_D, h_C denote the distances from the points M, D, C to line AB, respectively (see the first

diagram below).





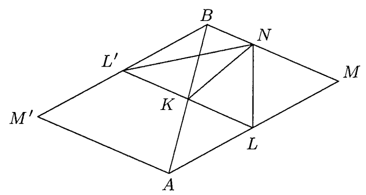
Then $h_M = \frac{1}{2}(h_D+h_C)$, and therefore (see the second diagram above)

$$\text{Area}\,ADK + \text{Area}\,BCK = \text{Area}\,AMB.$$

It follows that

$$\text{Area}\,ADL + \text{Area}\,BCN = \text{Area}\,KLMN.$$

Let $X \mapsto X'$ be the central symmetry with respect to the point K.



Since $B = A'$, $K = K'$ we have $\text{Area}\,ALK = \text{Area}\,BL'K$. It is also clear that $\text{Area}\,LKN = \text{Area}\,L'KN$. Therefore

$$\begin{aligned} \text{Area}\,ALK + \text{Area}\,BNK &= \text{Area}\,KNBL' > \text{Area}\,L'KN \\ &= \text{Area}\,LKN, \end{aligned}$$

that is,

$$\text{Area}\,LKN < \text{Area}\,ALK + \text{Area}\,BNK.$$

Similarly,

$$\text{Area}\,LMN < \text{Area}\,DLM + \text{Area}\,MNC.$$

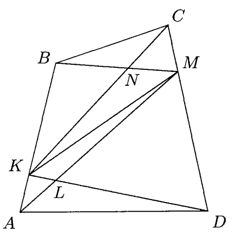
Consequently,

$$\begin{aligned}\text{Area}\,KLMN &< \text{Area}\,ALK + \text{Area}\,BNK \\ &\quad + \text{Area}\,DLM + \text{Area}\,MNC \\ &= \text{Area}\,ABCD - \text{Area}\,KLMN \\ &\quad - \text{Area}\,ADL - \text{Area}\,BCN \\ &= \text{Area}\,ABCD - 2\text{Area}\,KLMN.\end{aligned}$$

Therefore

$$\text{Area}\,KLMN < \frac{1}{3}\,\text{Area}\,ABCD.$$

(b) Denote the area of the quadrilateral $ABCD$ by S and the areas of the triangles ABC, BCD, CDA, DAB by b, c, d, a, respectively. Let $\alpha = m/(m+n)$.



Then $AK = \alpha AB$, and $CM = \alpha CD$. Let $S_1 = \text{Area}\,AMB$. Since

$$\text{Area}\,AMB = \alpha\,\text{Area}\,ADB + (1-\alpha)\,\text{Area}\,ACB,$$

then

$$S_1 = \alpha a + (1-\alpha)b.$$

Analogously, we obtain

$$S_2 = \text{Area}\,DKC = \alpha c + (1-\alpha)d.$$

Let $S_3 = \text{Area}\,AKMD$, then

$$\begin{aligned}S_3 &= \text{Area}\,AMK + \text{Area}\,MDA \\ &= \alpha\,\text{Area}\,AMB + (1-\alpha)\,\text{Area}\,CDA \\ &= \alpha S_1 + (1-\alpha)d.\end{aligned}$$

Analogously, we can prove that

$$S_4 = \text{Area}\,KBCM = \alpha S_2 + (1-\alpha)b.$$

We now express the area of $KLMN$ in terms of the introduced parameters. Since

$$\text{Area}\,KLMN = \text{Area}\,KLM + \text{Area}\,KMN,$$

it is sufficient for us to find expressions for Area KLM and Area KMN. Note that

$$\begin{aligned}\frac{\text{Area}\,DKM}{\text{Area}\,KLM} &= 1 + \frac{\text{Area}\,DLM}{\text{Area}\,KLM} = 1 + \frac{DL}{LK} \\ &= 1 + \frac{\text{Area}\,MDA}{\text{Area}\,AMK} = \frac{S_3}{\text{Area}\,AMK},\end{aligned}$$

and therefore,

$$\text{Area}\,KLM = \frac{\text{Area}\,DKM \cdot \text{Area}\,AMK}{S_3} = \alpha(1-\alpha)\frac{S_1S_2}{S_3}.$$

Analogously, we can prove that

$$\text{Area}\,KMN = \alpha(1-\alpha)\frac{S_1S_2}{S_4},$$

and, consequently,

$$\text{Area}\,KLMN = \alpha(1-\alpha)\frac{S_1S_2}{S_3S_4}\cdot S.$$

We must prove the following inequality

$$\text{Area}\,KLMN < \frac{\alpha(1-\alpha)}{1-\alpha+\alpha^2}\,\text{Area}\,ABCD.$$

This inequality is now equivalent to

$$(1-\alpha+\alpha^2)S_1S_2 < S_3S_4.$$

Substituting expressions for S_1, S_2, S_3, S_4 in terms of a, b, c, d, after obvious transformations we arrive at the inequality

$$\alpha(ab + cd - ac) + (1-\alpha)(b^2 + d^2 + bd - ad - bc) > 0.$$

Hence, it is sufficient to prove that the following inequalities hold

$$ab + cd - ac > 0 \quad \text{and} \quad b^2 + d^2 + bd > ad + bc.$$

In order to prove the first of these inequalities, note that

$$ab + cd \geq \min(a, c) \cdot (b + d) = \min(a, c) \cdot S > ac,$$

since $S > a$ and $S > c$. Furthermore,

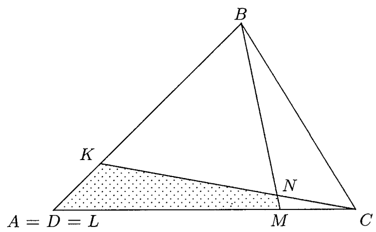
$$\begin{aligned} ad + bc &\leq \max(b, d) \cdot (a + c) = \max(b, d) \cdot S \\ &= \max(b, d) \cdot (b + d) < b^2 + d^2 + bd, \end{aligned}$$

from which the second inequality follows.

Remark. The estimate given in the problem for the area of quadrilateral $KLMN$ is optimal. The equation

$$\text{Area}\, KLMN = \frac{mn}{m^2 + mn + n^2}\, \text{Area}\, ABCD$$

holds for the degenerate quadrilateral shown in the diagram.

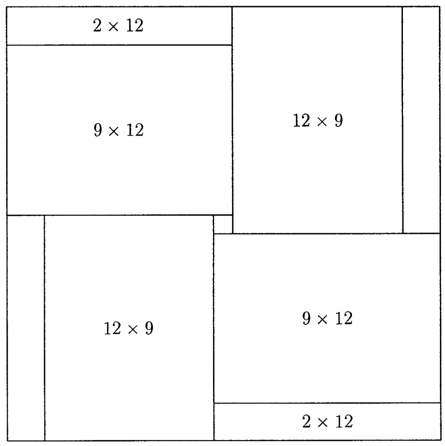


12. Answer: one 1×1 square is enough.

Paint columns 2, 3, 5, 6, 8, 9, 11, 12, 14, 15, 17, 18, 20, 21, 23 of the given 23×23 square. Then a total of 23×15 cells will be painted. Note that this number is odd. The 2×2 and 3×3 squares contain an even number of painted cells. Hence at least one 1×1 square must be used. And it is, indeed, possible to form a 23×23 square using only one 1×1 square.

In the diagram one way of forming a 23×23 square with only one 1×1 square is shown. The 9×12 rectangles must be further divided into 3×3 squares and 2×12 rectangles must be divided into 2×2

squares.



There are other possibilities.

13. (a) **Answer:** yes, for example, $a = 2 + \sqrt{2}$, $b = -\sqrt{2}$.

Let us first note that for an arbitrary n

$$(\sqrt{2} + 1)^n = p_n\sqrt{2} + q_n,$$

where p_n and q_n are positive integers and $q_n > 1$ for $n \geq 2$. This can be proved by an easy induction or directly by using the binomial theorem. Therefore $(\sqrt{2} + 1)^n$ is irrational. Now,

$$a^n + b^n = (\sqrt{2})^n((\sqrt{2} + 1)^n + (-1)^n).$$

If $n = 2k$, then the number

$$a^n + b^n = 2^k((\sqrt{2} + 1)^{2k} + 1),$$

is obviously irrational. If $n = 2k + 1$, then

$$\begin{aligned} a^n + b^n &= 2^k\sqrt{2}((\sqrt{2} + 1)^{2k+1} - 1) \\ &= 2^k(2p_n + (q_n - 1)\sqrt{2}) \end{aligned}$$

is also irrational for $k \geq 1$ since $q_n - 1 > 0$.

(b) **Answer:** no.

Suppose on the contrary that the numbers a and b satisfy the conditions of the problem. We first prove that $a \neq 0$ and $b \neq 0$. If, for example, $a = 0$ then b has to be different from zero and, since $b = (0+b^3)/(0+b^2)$, it follows that b is rational, which contradicts the conditions of the problem.

The identity

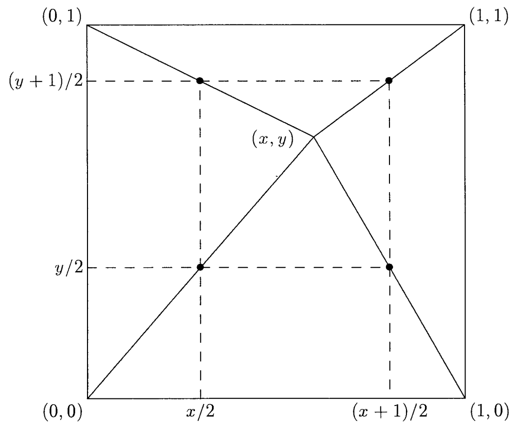
$$a^2b^2 = \frac{1}{2}(a^2+b^2)^2 - \frac{1}{2}(a^4+b^4)$$

implies that the number a^2b^2 is rational. Since $a^2b^2 \neq 0$, from the identity

$$a^5+b^5 = (a^3+b^3)(a^2+b^2) - a^2b^2(a+b)$$

it follows that the number $a+b$ is rational. This is a contradiction.

14. Let us introduce a rectangular system of coordinates on the ceiling, with the origin at one of its corners and the axes going along the corresponding edges. Take 1 metre as the unit of length.



Let (x, y) be the coordinates of the initial position of the spider.

Consider the set of points

$$\Pi_k = \left\{ \left(\frac{x+i}{2^k}, \frac{y+j}{2^k} \right), \quad i, j \text{ are integers }, \quad 0 \le i, j < 2^k \right\}.$$

This set gives us a network of cells whose sides have length equal to 2^{-k}. Clearly, Π_0 consists of the single point (x, y)). The four points of Π_1 are shown in the diagram above.

We see that in jumping from a point (x, y) the spider can change the first coordinate x of its location into either $x/2$ or $(x+1)/2$ and independently it can change the second coordinate y into either $y/2$ or $(y+1)/2$. Let us show that the spider can jump to any point of the set Π_k after k jumps. We prove this by induction. For $k = 0$ this is clear. Suppose that the spider can jump on any point of Π_k after k jumps. Take an arbitrary point

$$A = \left(\frac{x+i}{2^{k+1}}, \frac{y+j}{2^{k+1}} \right)$$

of Π_{k+1}. We have to find a point $B = (x_1, y_1)$ of Π_k from which the spider can jump to A. Take

$$x_1 = \frac{x+i}{2^k}, \quad \text{if } i < 2^k,$$

and

$$x_1 = \frac{x+i}{2^k} - 1, \quad \text{if } i \ge 2^k,$$

and similarly

$$y_1 = \frac{y+j}{2^k}, \quad \text{if } j < 2^k,$$

and

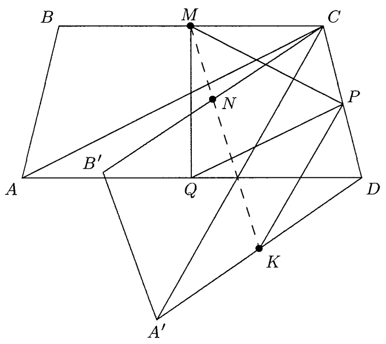
$$y_1 = \frac{y+j}{2^k} - 1, \quad \text{if } j \ge 2^k.$$

The point B with these coordinates lies in Π_k, and the spider can jump from it to A.

It is now clear that for any point C on the ceiling there exists a point in Π_k, whose distance from C is not greater than $\sqrt{2} \cdot 2^{-k}$. Since for $k = 8$ we have $\sqrt{2} \cdot 2^{-8} < \frac{1}{100}$, this solves the problem.

15. First solution. Let us denote the angle of the rotation by α, and suppose that $0 \le \alpha \le 360°$. Let K, M, N be the midpoints of the segments $A'D, BC, B'C$, respectively and let P and Q be the

midpoints of the side CD and the base AD respectively. In the following diagram one possible arrangement of the points is shown.



Then MQ is a common perpendicular to BC and AD, and $MP = PQ$. A homothety with centre D and scale factor $\frac{1}{2}$ maps the points A, C, A' onto Q, P, K, respectively. Therefore,

$$PQ = \frac{1}{2}AC = \frac{1}{2}A'C = PK$$

and

$$\angle QPK = \angle ACA' = \alpha.$$

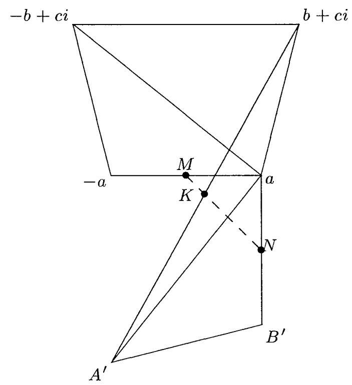
Since $MP = PQ = PK$, the points M, Q, K lie on a circle with centre P and $\angle QMK = \frac{1}{2}\angle QPK = \frac{\alpha}{2}$. Let us prove that $\angle QMN = \frac{\alpha}{2}$. Note that QM is a tangent line to the circle of radius MC with centre C. Since $NC = MC$, the point N is on this circle, and by the theorem above the angle between a chord and a tangent we have that $\angle QMN = \frac{1}{2}\angle MCN = \frac{\alpha}{2}$.

Thus $\angle QMK = \angle QMN$. Moreover, the points N and K are on the same side of the line MQ. Hence the points M, N, K lie on the same line.

But the solution is not over yet. We considered only one possible arrangement of points, but others are possible. For example, what would happen if the angle of rotation were bigger? The last stage of a correct solution would be to check thoroughly that our arguments

do not depend on the diagram. And indeed they do not. We omit these considerations here, but instead we present a second solution for which such considerations are not needed.

Second solution. Place our trapezium on the complex plane, so that the point M coincides with the origin and the points B and C are represented by the real numbers $-a$ and a, respectively, where $a > 0$. Then A and D will be represented by the complex numbers $-b + ci$ and $b + ci$, respectively, where $b > 0$ (see diagram).



Let ϵ be the complex number such that multiplication by ϵ rotates the plane through the angle α. Then the point N will be represented by the complex number

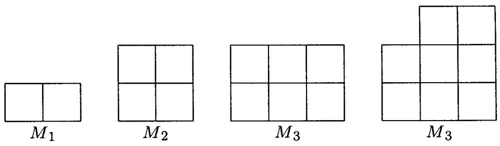
$$u = -a\epsilon + a = a(1 - \epsilon).$$

The point A' will be represented by the complex number $(-a - b + ci)\epsilon + a$, and the point K will be represented by the complex number

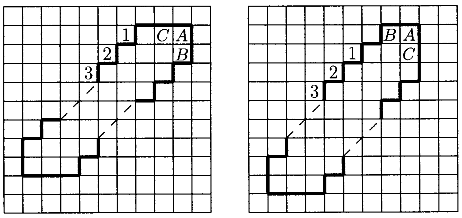
$$\begin{aligned} v &= \frac{1}{2}\left[((-a - b + ci)\epsilon + a) + b + ci\right] \\ &= \frac{1}{2}\left[a(1-\epsilon) + b(1-\epsilon) + ic(1+\epsilon)\right]. \end{aligned}$$

The statement of the problem is true if v is a real multiple of u. This is indeed the case, since $1 + \epsilon$ is perpendicular to $1 - \epsilon$, and therefore $i(1 + \epsilon)$ must be a real multiple of $1 - \epsilon$.

16. The figures M_1, M_2, M_3, M_4 in the following diagram



can be cut into 2×1 rectangles in $1, 2, 3, 4$ ways respectively. The diagrams below



show polygons $M_n, n \geq 5$, which can be divided into 2×1 rectangles in exactly n ways for n odd and even, respectively. To prove this, we notice that if we delete the cells, marked A and C on the diagram from M_n, the remaining polygon can be divided into 2×1 rectangles in a unique way. If we delete from M_n the cells marked A and B, then we obtain M_{n-1}. Thus, if we denote by $|M_n|$ the number of different ways in which the polygon M_n can be divided into 2×1 rectangles, we obtain the following equation

$$|M_n| = 1 + |M_{n-1}|, \quad n \geq 5,$$

from which the result follows.

17. Answer: $n = 7$.

The given equation leads to the inequality

$$1989 \leq \frac{10^n}{x} < 1990,$$

which is equivalent to

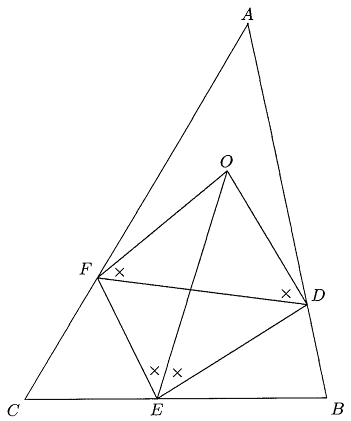
$$\frac{10^n}{1990} < x < \frac{10^n}{1989}$$

and to

$$10^n \cdot 0,0005025\ldots < x \leq 10^n \cdot 0,0005027\ldots.$$

This inequality has solutions in the set of integers only for $n \geq 7$. For $n = 7$ we have two solutions: $x_1 = 5026$ and $x_2 = 5027$.

18.



$$\angle DEF = 180^\circ - (180^\circ - 2\angle B) - (180^\circ - 2\angle C) = 180^\circ - 2\angle A,$$

so $\angle A$ is acute. Consequently, the centre O of the circle circumscribed around triangle ADF and the vertex A lie on the same side of the segment FD, and therefore,

$$\angle DOF = 2\angle A = 180^\circ - \angle DEF.$$

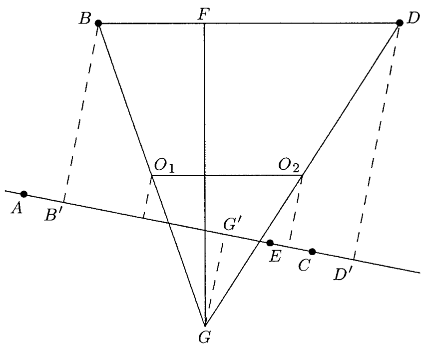
It follows that the points O, D, E, F lie on the same circle, and hence

$$\angle DEO = \angle DFO = \angle FDO = \angle FEO,$$

so that EO is the bisectrix of $\angle DEF$.

19. Let E and F be the points common to the given spheres through which the segments AC and BD pass and let O_1 and O_2 be the

centres of the spheres.



Then the point G, which is the reflection of the point F in the line O_1O_2, is also common to the 2 spheres. BD is parallel to O_1O_2, so O_1O_2 is perpendicular to FG, then BG and DG are diameters of these spheres.

Let B', D', G' be the projections of the points B, D, G onto the line AC. Since the midpoint O_1 of the segment BG projects onto the midpoint of the chord AE, which coincides with the midpoint of the segment $B'G'$, we have the equality $AB' = EG'$. Similarly, one can prove the equality $CD' = EG'$. From these equalities, it follows that $AB' = CD'$.

20. Answer: Yes, it is.

Without loss of generality, assume that the broken line (path) $A_1A_2 \ldots A_{n+1}$, where $A = A_1$, $B = A_{n+1}$, lies in a certain vertical plane and each of its segments makes an angle of 45° with the horizontal line AB. Let us define the rules according to which the tourists should move:

- Both tourists travel with equal constant speed at all times;
- If at any time one of the tourists cannot continue to move in the same direction because the other is forced to lose or gain altitude, then the tourist turns and goes back with the same speed.

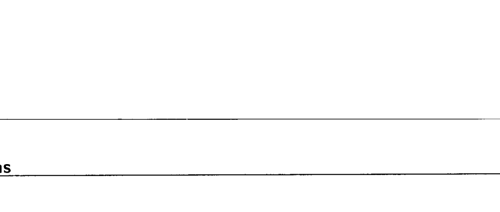
We shall now formulate a number of observations which follow from the rules.

(a) If at any time just one of the tourists reaches a vertex of the broken line, then he/she continues to move in the same direction (or to be more precise we should say that the projection onto the horizontal line AB moves in the same direction) while the other must turn back.

(b) If both tourists reach vertices at the same time then they both continue to move in their directions. Thus the tourists cannot both turn back at the same time, and if they meet they are forced to pass each other. This can only be at a vertex.

We say that the tourists are in state (i, j) if the first tourist is on the slope A_iA_{i+1} and the second is on the slope A_jA_{j+1}. The motion of the tourists can now be characterised as a sequence of states.

For example, the motion of the tourists on the mountain chain given in the following diagram



will be given by a sequence:

$$\begin{aligned}(1,6) &\to (2,5) \to (3,5) \to (3,6) \to (2,6)\\ &\to (1,5) \to (1,4) \to (2,4) \to (3,4)\\ &\to (4,3) \to (4,2) \to (4,1) \to (5,1)\\ &\to (6,2) \to (6,3) \to (5,3) \to (5,2) \to (6,1).\end{aligned}$$

We say that the state (k, l) is a neighbour of (i, j), if the tourists can move to (i, j) from (k, l), by moving in one or the other direction. Clearly, if (k, l) is a neighbour of (i, j), then (i, j) is a neighbour of (k, l). Every state, other than $(1, n)$ and $(n, 1)$, has exactly two neighbours, while $(1, n)$ and $(n, 1)$ have only one neighbour each.

Let (i,j) and (k,l) be neighbours. We noticed earlier that the tourists cannot simultaneously turn back. Therefore they cannot return to the state (i,j) if they have just moved from (i,j) to (k,l).

We claim that no state can appear twice. Suppose on the contrary, that (k,l) is the first state that occurs twice. Clearly, it cannot be $(1,n)$ or $(n,1)$ since if it were its only neighbour would appear twice before it does. Let (i,j) and (m,n) be the neighbours of (k,l). When (k,l) occurs for the first time, the tourists move either

$$(i,j) \to (k,l) \to (m,n) \to (p,q)$$

or

$$(m,n) \to (k,l) \to (i,j) \to (p,q).$$

In either case both neighbours of (k,l) appear before (k,l) appears for the second time. And in both cases the state (p,q) is different from (k,l). When (k,l) appears for the second time, it will be approached either from (i,j) or from (m,n). Therefore one of the neighbours of (k,l) will appear for the second time before (k,l) does.

We have n^2 possible states and no state can occur twice. For every state other than $(n,1)$, the tourists must move and they can never return. Therefore starting from $(1,n)$ they must finish at $(n,1)$.

21. Answer: 2.

For arbitrary positive x,y,z satisfying the equation $xyz(x+y+z) = 1$, we have

$$(x+y)(y+z) = (x+y+z)y + xz = \frac{1}{xz} + xz \geq 2.$$

This inequality becomes an equality, when for example, $x = z = 1$ and $y = \sqrt{2} - 1$.

22. Let $A_1, A_2, \ldots, A_m$ be the vertices of the polyhedron. To start with, we place an arrow arbitrarily on each edge of the polyhedron. Suppose that the vertex A_i is pointed at by n_i arrows. We say that a vertex is odd, if it is pointed at by an odd number of arrows. The number of odd vertices is even, since the sum $n_1 + n_2 + \ldots + n_m$ is equal to the number of edges of the polyhedron, which is even by hypothesis.

If odd vertices exist, let us select a pair of such vertices, say A_i and A_j. We travel from one to the other along the edges of the polyhedron, changing the directions of all arrows through which we pass. As a result the vertices A_i and A_j cease to be odd, while the character of the rest of the is unchanged. Applying this operation as many times, as necessary, we eliminate all odd vertices.

23. Answer: No, there are no such functions.

If there were such a function, then among its values

$$f(2), f(3)\ldots, f(n), \ldots$$

there would be a minimal one, say $f(n_0)$, $n_0 > 1$. Note that

$$f(n_0+1) \geq f(n_0) = f(f(n_0-1)) + f(f(n_0+1)) \geq 1+1 > 1,$$

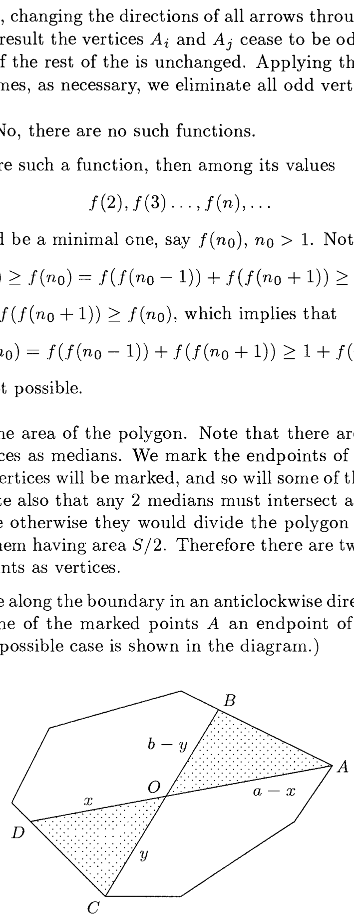
Therefore, $f(f(n_0+1)) \geq f(n_0)$, which implies that

$$f(n_0) = f(f(n_0-1)) + f(f(n_0+1)) \geq 1 + f(n_0),$$

which is not possible.

24. Let S be the area of the polygon. Note that there are at least as many vertices as medians. We mark the endpoints of all medians. All of the vertices will be marked, and so will some of the boundary points. Note also that any 2 medians must intersect at an interior point, since otherwise they would divide the polygon into 3 parts with 2 of them having area $S/2$. Therefore there are twice as many marked points as vertices.

Let us move along the boundary in an anticlockwise direction, starting from one of the marked points A an endpoint of the median AD. (One possible case is shown in the diagram.)



If the endpoint B of the median BC is the marked point following A, then C will be the marked point following D. Indeed, no other

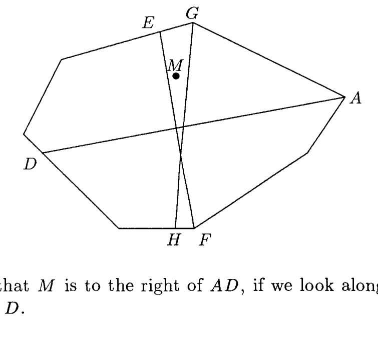
endpoint K of a third median can lie between C and D, since the other end of this median, which has to intersect medians AD and BC, would lie between A and B. Therefore any two neighbouring medians AD and BC form a butterfly, by which we understand a figure consisting of two triangles AOB and COD, where O is the point of intersection of the medians AD and BC. Note that the areas of the two triangles AOB and COD of the butterfly are the same, because each of them after being added to the area within angle BOD becomes equal to $S/2$. The area of the butterfly $ABOCD$ is

$$\begin{aligned}\text{Area}\,ABOCD &= \frac{1}{2}xy\sin\alpha + \frac{1}{2}(a-x)(b-y)\sin\alpha \\ &= \frac{ab\sin\alpha}{4} + (\frac{a}{2}-x)(\frac{b}{2}-y)\sin\alpha,\end{aligned}$$

where $a = AD$, $b = BC$, $\angle AOB = \angle COD = \alpha$, $x = OD$, $y = OC$. This area is less than $\alpha/4$ because $a \le 1$, $b \le 1$, $0 < \sin\alpha < \alpha$, and the numbers $a/2 - x$, $b/2 - y$ cannot be either both positive or both negative (for if they were both positive $\text{Area}\,AOB < \text{Area}\,COD$, and if they were both negative then $\text{Area}\,AOB > \text{Area}\,COD$).

Moving further along the boundary from point B to point D, we obtain all butterflies formed by neighbouring medians. The sum of the areas of these butterflies, according to what was proved earlier, is less than one quarter of the sum of the corresponding angles, which is less than $\pi/4$.

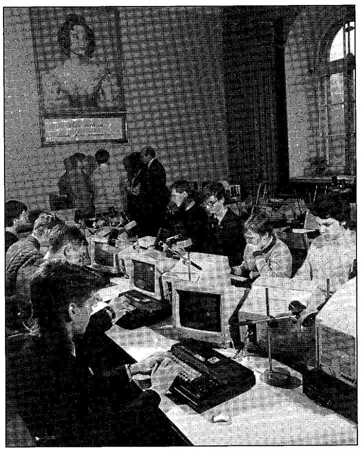
It remains only to prove that such butterflies cover the whole area of the polygon, for then $S < \pi/4$ will follow. Let M be an arbitrary point inside the polygon. Clearly we can assume that M does not belong to any of the medians.



Suppose that M is to the right of AD, if we look along AD from A toward D.

From A we move anticlockwise until we first come to a marked point E, the endpoint of some median EF, such that M is to the left of EF if we look along EF from E toward F. Since for the previous median GH it was still on the right, the two medians EF and GH form a butterfly which includes M. Thus the proof is now complete.

(Left to right) Yu Nesterenko, A Astrelin, E Pankrat'ev, R Freivalds, A Slinko under the portrait of Ada Augusta, Countess of Lovelace (1815–1852). Ada Augusta, daughter of the poet Byron, was the world's first debugger of computer programs while working with Charles Babbage.



Computer Tournament participants at work.

COMPUTER TOURNAMENT 1989

This tournament was organized after the USSR Olympiad in Informatics was established, and the Jury of the Mathematical Olympiad observed a significant shift of mathematically gifted students from the Math Olympiad to the Olympiad in Informatics. Unfortunately both olympiads were scheduled approximately at the same time during the short spring break in school studies. There were no doubts that all mathematically gifted kids like computer very much, and some of them had a really difficult choice between the two olympiads.

At the initiative of the author, the Jury of the USSR Mathematical Olympiad, set up in 1989 the Computer Tournament of the Mathematical Olympiad. Due to this initiative, I was the Chairman of the Jury of this Tournament. The Jury also included professional computer scientists V Rozhdestvenski and E Pankrat'ev from the Moscow State University, and E Tyrtyshnikov from the Institute of Computational Mathematics of the USSR Academy of Sciences. The former participant of IMO A Astrelin was a key member of our team as he could instantly and without mistakes write a very complicated programme in almost all programming languges.

We were so lucky that the Olympiad of 1989 took place in Riga in the capital of Latvia for at least two reasons. Firstly, the school in Latvia was well-equipped with computers and secondly, the last but not the least, that we had a lot of help from the the top Latvian computer scientists, and Rusins Freivalds in the first place, who was the Chairman of the Jury of the Mathematical Olympiad, and from many other people including school-teachers of those schools where the Computer Tournament took place.

The Computer Tournament was a separate competition for the same students. It gave them an opportunity to compete in solving problems with the use of a computer without going to another olympiad. About 80% of the participants took part in it with considerable interest. Our idea was to compose problems which were mathematical and such that their solutions needed the use of a computer on some stage. Such a solution usually have two components, theoretical and computational. A student had, first to solve certain mathematical problem, and second to compute three test examples in accordance with what she/he proved. The test numbers were unknown to the students until they had finished their work. If all three answers were correct, the algorithm was considered to be programmed correctly. If only one or two answers were correct students also got some points. Time was not a prime parameter but there was a time limit for every question. Since the computers were so slow

students had to write really smart algorithms to be able to compute the third test example.

PROBLEMS

1. Given a pair of integers (a, b), we are allowed to add these two numbers and to write the sum $a+b$ instead of a or b to obtain one the pairs $(a, a+b)$ or $(a+b, b)$. Design an algorithm that computes the least possible number of such transformations necessary to obtain a pair containing the number $N < 2000$ starting with the pair $(1, 1)$. Write down the computer program and enter it into the computer.

(Test numbers: $N = 700$; 800; 1989.)
(V Rozdestvenski, Moscow)

2. The sequence (a_n) is given by $a_1 = 1/e$ and $a_n = 1 - na_{n-1}$. Compute a_n for $n \leq 50$ correct to 4 decimal places.

(Test numbers: $N = 10; 20; 30$.)
(E Tyrtyshnikov, Moscow)

3. A sequence (x_n) is said to be periodic if for some M and T the equation $x_n = x_{n+T}$ holds for every $n > M$. The least numbers M and T with this property is said to be the length of the irregular part and the period of the sequence, respectively.

Let $f(x)$ be a function mapping the set of positive integers into itself. A sequence (x_n) is defined by its first term x_1 and the equation $x_{n+1} = f(x_n)$ for $n > 1$.

(a) Prove that, if f is bounded, the sequence (x_n) is periodic.

(b) Find an algorithm that computes the length of the irregular part M and the period T for the given sequence with the property that the number of comparisons required is a linear function of the maximum of these two numbers.

(c) Program your algorithm into the computer for the function $f(x)$, whose value at x is the remainder on dividing $g(x)$ by 10^8, where

$$g(x) = \begin{cases} x/3 & \text{if } x = 3k, \\ 2(x-1)/3 & \text{if } x = 3k+1, \\ (13x-2)/3 & \text{if } x = 3k+2 \end{cases}$$

(Test numbers: $x_1 = 35$; 666; 12345678.)
(Ju Nesterenko, A Slinko; Moscow)

SOLUTIONS

1. Let (M, N) be a pair of numbers, which can be obtained from the pair $(1,1)$. The moves needed to get from $(1,1)$ to (M, N) can be reconstructed since if $a > b$ the pair (a, b) can only be obtained from the pair $(a-b, b)$. For any particular M starting with (M, N) and going backwards if we reach $(1,1)$ we can count the number of transformations required.

   Since (M, N) and $(N-M, N)$ give the same pair $(N-M, M)$ after the first step of the reconstruction, it is sufficient to test only those numbers M between 1 and $[N/2]$.

   If the greatest common divisor of M and N is not 1, we will never get the pair $(1,1)$ during the reconstruction. Therefore, it pays to find the prime factorization of N first.

   Answers for test examples: 15, 16, 18.

2. The number a_1 has no exact representation on a computer and so must be represented by $b_1 = a_1 + \epsilon_1$. Even without rounding errors, computing $a_2, a_3, \ldots$, subsequently we will obtain $b_2, b_3, \ldots$, where $b_n = a_n + \epsilon_n$, and $|\epsilon_n| = n!|\epsilon_1|$. The initial error will be multiplied by $n!$. Obviously, such a computation is inadequate even if we compute with multiple precision.

   Some theoretical deliberations are in order. Starting with the well-known formula

$$\frac{1}{e} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots$$

   it is easy to show by induction that

$$\begin{aligned} a_n &= n!\left(\frac{1}{(n+1)!} - \frac{1}{(n+2)!} + \frac{1}{(n+3)!} - \frac{1}{(n+4)!} + \ldots\right) \\ &= \frac{1}{n+1} - \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} - \ldots. \end{aligned}$$

   Pairinging the summands, starting from the first term, we get $a_n > 0$, and pairing them, starting from the second term, we get $a_n < \frac{1}{n+1}$. This approximation allows us to carry out the reverse computation by using the formula

$$a_{n-1} = \frac{1 - a_n}{n}.$$

   and starting from, say $b_{100} = 0$. The idea is that in this case the initial error ϵ_{100}, which is less than 0.01, will be diminished at each step being divided by n and in due course will become negligible. The rounding errors on most personal computers do not exceed 10^{-7}.

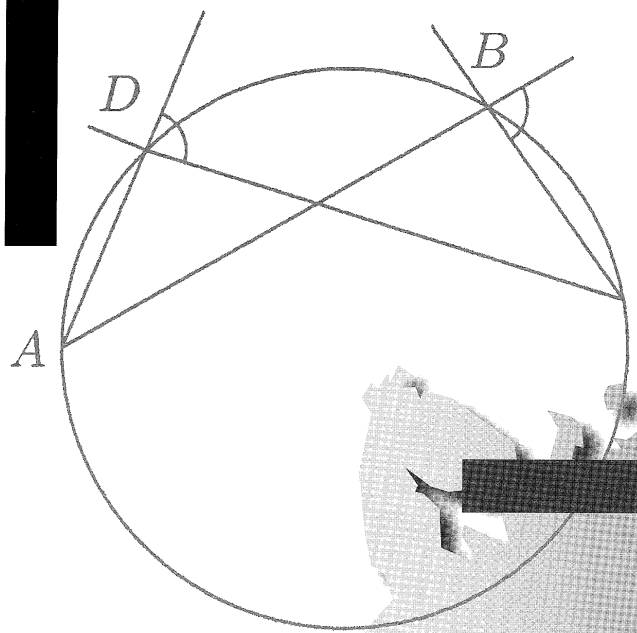
   Answers for test examples: 0.0838...; 0.0455...; 0.0312....

3. First, we compare the numbers x_n and x_{2n} for $n = 1, 2, \ldots$ until we finf a pair which coincide. Let k be the smallest natural number for which $x_k = x_{2k}$. Clearly, k is the smallest multiple of T, which is greater than M. Thus

$$M < k < M + T.$$

   This requires no more than $M + T$ comparisons. Next, we compare the numbers x_k and x_{k+i} for $i = 1, 2, \ldots$ and find the smallest number i such that $x_k = x_{k+i}$. This will be equal to the period T. This requires no more than T comparisons. Finally, we compare the numbers x_j and x_{j+T} for $j = 1, 2, \ldots$. If j is the smallest number such that $x_j = x_{j+T}$, then $M = j - 1$. We need M comparisons here. In all we need no more than $2(M + T)$ comparisons.

   Answers for test examples: $(M, T) = (4, 14)$, $(8, 14)$, $(125, 14)$.



$$3^{2n+1} - 2^{2n+1} - 6^n = ($$

USSR OLYMPIAD 1990

9 FORM[2]

First day

1. Prove that for an arbitrary t the inequality $t^4 - t + \frac{1}{2} > 0$ holds.
(I Voronovich, Minsk)

2. A line passing through the midpoints of two opposite sides of a convex quadrilateral forms equal angles with both diagonals. Prove that the lengths of the diagonals are equal.
(A Andzans, Riga)

3. In a Senate there are 30 senators. For each pair of senators they are either friends or enemies. Every senator has 6 enemies. Any 3 senators form a commission. Find the total number of commissions, whose members are either all friends or all enemies of each other.
(D Fomin, St. Petersburg)

4. (a) Does there exist a rectangle that can be cut into 15 congruent polygons, which are not rectangles?

(b) Does there exist a square with this property?

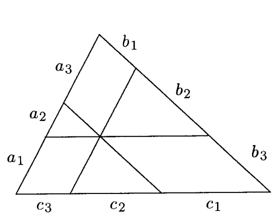
(S Eliseev)

[2]Note that numbering of forms has changed due to a reform in which one introductory school year was introduced. The students were approximately aged 15 as in 8 form in 1989.

9 FORM

Second day

5. Through an arbitrary point inside a triangle, three lines parallel to the sides of the triangle are drawn. They divide the sides into segments of lengths a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, as shown below.



Prove that $a_1b_1c_1 = a_2b_2c_2 = a_3b_3c_3$.

(B Chinik, Kishinev)

6. Two boys are playing a game. The first boy chooses three arbitrary non-zero numbers. The second substitutes them for the stars in the equation $*x^2 + *x + * = 0$ in an arbitrary order. The first player is considered the winner if the resulting quadratic equation has two distinct rational roots. Prove that he can win.

(A Berzinsh, Riga)

7. Find the maximum value of the expression

$$|\ldots||x_1 - x_2| - x_3| - \ldots - x_{1990}|,$$

where $x_1, x_2, \ldots, x_{1990}$ are distinct natural numbers between 1 and 1990.

(O Bogopol'sky, Novosibirsk)

8. An equilateral triangle, whose sides are of length n, is cut into n^2 equilateral triangles whose sides are of length 1 by lines parallel to the sides. A broken line is drawn so that its edges go along the lines of this lattice. The broken line is not closed, and it passes through all vertices of the lattice exactly once. Prove that at least n pairs of neighbouring edges of the broken line have an acute angle between them.

(A Berzinsh, Riga)

10 FORM

First day

9. Is it possible to colour the cells of a 1990×1990 square grid in black and white, so that cells symmetric with respect to the centre of the grid have different colours, and so that in each column and each row half of the cells are black and half of the cells are white?

(N Agahanov, Moscow)

10. Let $a_1, a_2, \ldots, a_n$ be positive numbers, whose sum is 1. Prove that

$$\frac{a_1^2}{a_1 + a_2} + \frac{a_2^2}{a_2 + a_3} + \ldots + \frac{a_{n-1}^2}{a_{n-1} + a_n} + \frac{a_n^2}{a_n + a_1} \geq \frac{1}{2}.$$

(D Tereshin, Moscow)

11. On the side AB of a convex quadrilateral $ABCD$ a point E, different from the vertices, is chosen. The segments AC and DE intersect at a point F. Prove that the circles circumscribed about the triangles ABC, CDF and BDE have a common point.

(L Kuptzov, Moscow)

12. Two grasshoppers are sitting at the endpoints of the segment $[0, 1]$. A finite number of points of the segment (at least one) are marked, dividing $[0, 1]$ into a finite number of intervals.

A grasshopper can choose any one of the marked points and jump over it to the point which is the same distance from the chosen marked point as his previous location was, provided that this point also belongs to the segment $[0, 1]$. In one move the grasshoppers either jump simultaneously according to this rule, or else one of them jumps and the other stays where it is.

What is the smallest number of moves needed to ensure that the grasshoppers occupy locations in the same interval, that is, with no marked points between them.

(S Konyagin, Moscow)

10 FORM

Second day

13. Find all integer solutions of the equation

$$\left[\frac{x}{1!}\right]+\left[\frac{x}{2!}\right]+\ldots+\left[\frac{x}{10!}\right]=1001.$$

(S Reznichenko, Moscow)

14. On the sides A_1A_2 and A_2A_3 of a regular $2n$-gon $A_1A_2\ldots A_{2n}$ points K and N are chosen, respectively, so that $\angle KA_{n+2}N=\pi/2n$. Prove that NA_{n+2} is the bisector of the angle KNA_3.

(N Agahanov, D Tereshin, Moscow; D Fomin, St.Petersburg)

15. In a convex polygon all diagonals are drawn. Each side and each diagonal is coloured in one of k colours so that no closed broken line, whose vertices are also vertices of the polygon, is coloured with one colour. What is the largest number of vertices for which this is possible?

(A Andzans, Riga; D Flaas, Novosibirsk)

16. In the plane a point A_0 and n vectors $\vec{a}_1,\ldots,\vec{a}_n$ are given. The sum of those vectors is zero. Each permutation $\vec{a}_{i_1},\ldots,\vec{a}_{i_n}$ of these vectors defines a set of points $A_1,A_2,\ldots,A_n=A_0$ such that

$$\vec{a}_{i_1}=\overrightarrow{A_0A_1},\vec{a}_{i_2}=\overrightarrow{A_1A_2},\ldots,\vec{a}_{i_n}=\overrightarrow{A_{n-1}A_n}.$$

Prove that there exists a permutation for which the points $A_1,\ldots,$ A_{n-1} are located inside or on the sides of some $60°$ angle with its vertex at the point A_0.

(S Avgustinovich, S Sevastyanov, Novosibirsk)

11 FORM

First day

17. Two common tangents of two intersecting circles meet at a point A. Let B be a point of intersection of the two circles, and C and D be the points at which one of the tangents touches the circles. Prove that the line AB is tangent to the circle passing through B, C and D.

(I Sharygin, Moscow)

18. Suppose we are given 1990 piles, consisting of $1,2,\ldots,1990$ stones, respectively. In one move it is possible to choose several (maybe one) piles and remove from each chosen pile an equal number of stones. What is the least possible number of moves needed to remove all stones from all piles?

(N Agahanov, Moscow)

19. All coefficients of a quadratic polynomial $f(x)=ax^2+bx+c$ are positive, and $a+b+c=1$. Prove that the inequality

$$f(x_1)\cdot\ldots\cdot f(x_n)\geq 1$$

holds for all positive numbers $x_1,\ldots,x_n$, satisfying $x_1\cdot\ldots\cdot x_n=1$.

(D Fomin, St Petersburg)

20. A cube, with the edges of length 100, is composed of one million unit cubes whose edges form a three-dimensional grid. Any 3 unit edges with a common vertex is called a frame. Is it possible to decompose the grid into frames with no common edges?

(A Berzinsh, Riga)

11 FORM

Second day

21. For which positive integers n is the number $3^{2n+1} - 2^{2n+1} - 6^n$ composite?

(D Fomin, St Petersburg)

22. A pair of edges of a tetrahedron are called opposite if they do not have common vertices. Let d be the minimal distance between two opposite edges of a tetrahedron, and h be its minimal altitude. Prove that $2d > h$.

(A Skopenkov, Moscow)

23. The following equation with erased coefficients is written on a blackboard:

$$x^3 + \dots x^2 + \dots x + \dots = 0.$$

Two players are playing a game. In one move the first player chooses a number and the second player puts it instead of dots into one of the vacant places. After three moves the game is over. (Note that the first player chooses a number in each of the three moves.) Is it possible for the first player to choose three numbers that will secure three distinct integer roots for the equation, no matter how the second player plays?

(A Berzinsh, Riga)

24. We are given $4m$ coins, among which exactly half of the coins are counterfeit. All genuine coins have equal weights, all counterfeit coins also have equal weights but a counterfeit coin is lighter than a genuine one. Show how to determine all counterfeit coins in no more than $3m$ weighings, using a balance without weights?

(L Kurlyandchik, D Fomin, St Petersburg)

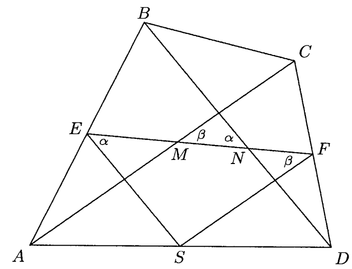
SOLUTIONS

1. Indeed,

$$\begin{aligned} t^4 - t + \frac{1}{2} &= \left(t^4 - t^2 + \frac{1}{4}\right) + \left(t^2 - t + \frac{1}{4}\right) \\ &= \left(t^2 - \frac{1}{2}\right)^2 + \left(t - \frac{1}{2}\right)^2 > 0, \end{aligned}$$

The inequality holds since both summands are nonnegative and cannot be equal to zero simultaneously.

2. Let E and F be the midpoints of AB and CD, and let also M and N be the points of intersection of EF with the diagonals AC and BD, respectively. There are two slightly different cases as the point of intersection of the diagonals can be above or below EF. We will consider only the first case.

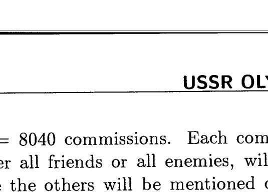


Denote the midpoint of AD by S. Since AC is parallel to SF, we have $\angle CMF = \angle EFS$ and $\angle BNE = \angle FES$. It is known that $\angle CMF = \angle BNE$, therefore $\angle EFS = \angle FES$ and the triangle ESF is isosceles with $SF = SE$. But SF and SE are the midlines in the triangles ACD and DBA, hence $AC = DB$.

3. Let x denote the number of commissions sought, and let y denote the number of commissions, among whose members are friends as well as enemies. Then $x + y = \frac{30\times29\times28}{2\times3} = 4060$. If each senator writes down a list of all his commissions, in which the other two members are both his friends or both his enemies, he obtains a list of $\frac{6\times5}{2} + \frac{23\times22}{2} = 268$ commissions. If the lists are combined there

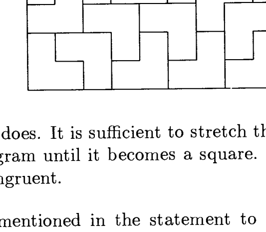
will be $30 \times 268 = 8040$ commissions. Each commission, whose members are either all friends or all enemies, will be mentioned three times, while the others will be mentioned only once each. Hence $3x + y = 8040$. Solving the pair of linear equations in x and y, we find that $x = 1990$.

4. (a) Yes, there does: see the diagram.



(b) Yes, there does. It is sufficient to stretch the rectangle shown in the diagram until it becomes a square. The polygons will remain congruent.

5. The segments mentioned in the statement to be proved are the sides of the three similar triangles shaded in the diagram.



From the similarity of these triangles it follows that

$$\frac{a_1}{a_2} = \frac{b_3}{b_1} \text{ and } \frac{b_3}{b_2} = \frac{c_2}{c_1}.$$

Therefore

$$a_1b_1c_1 = \frac{a_2b_3}{b_1} \cdot b_1c_1 = a_2b_3c_1 = a_2 \cdot \frac{b_2c_2}{c_1} \cdot c_1 = a_2b_2c_2.$$

and the first equality is proved. The proof of the second is similar.

6. It is sufficient for the first player to choose the numbers $1, 2, -3$ or any other distinct non-zero rational numbers a, b, c, whose sum is zero. Then, regardless of the choice of the second player, the number 1 will be a root of the polynomial and the other root will also be rational. The other root of $ax^2 + bx + c = 0$ will be c/a, which is different from 1 as $a \neq c$.

7. Answer: 1989.

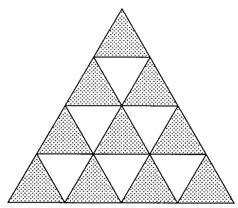
Since for $x \geq 0$, $y \geq 0$ the inequality $|x - y| \leq \max\{x, y\}$ holds, by induction

$$|\ldots||x_1 - x_2| - x_3| - \ldots - x_n| \leq \max\{x_1, x_2, \ldots, x_n\}$$

and hence the maximum value of this expression is not greater than 1990. But it cannot be 1990, since its parity equals the parity of the sum $x_1 + x_2 + \ldots + x_{1990} = 1 + 2 + \ldots + 1990 = 995 \times 1991$. However, 1989 can be achieved as the following example shows

$$\begin{aligned} &||||||\ldots||||\ldots||||2 - 4| - 5| - 3| - \ldots \\ &\ldots - (4k+2)| - (4k+4)| - (4k+5)| - (4k+3)| - \ldots \\ &\ldots - 1986| - 1988| - 1989| - 1987| - 1990| - 1| \\ &\qquad = ||0 - 1990| - 1| = 1989. \end{aligned}$$

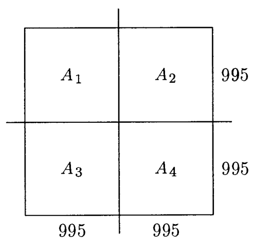
8. There are $(n+1)(n+2)/2$ vertices of the lattice, hence the broken line has $(n+1)(n+2)/2 - 1 = (n^2 + 3n)/2$ edges. Let us shade the triangles of the lattice in the pattern shown in the diagram.



In all we obtain $1 + 2 + \ldots + n = (n^2 + n)/2$ shaded triangles. Note that each edge of the broken line goes along the side of exactly one shaded triangle. The difference between the number of edges and

the number of shaded triangles is equal to $(n^2+3n)/2-(n^2+n)/2 = n$, thus at least n shaded triangles contain a pair of edges of the broken line which must be adjacent. The angle between the edges of these pairs is $60°$ and hence acute.

9. No, it is impossible. Consider an arbitrary colouring of the 1990×1990 grid, with symmetric cells painted in different colours. Let us write $+1$ in each black cell and -1 in each white one. Divide the grid into 4 squares A_1, A_2, A_3, A_4 of dimensions 995×995 as shown in the diagram.



Each of these squares contains an odd number of cells so the sum of the numbers inside each square is not zero. Since symmetric cells contain opposite numbers, the sum of the numbers in squares A_1 and A_4, as well as the sum of the numbers in squares A_2 and A_3, is equal to zero. Thus the sum of the numbers in 2 of the 4 squares is positive, and these squares are not symmetric with respect to the centre. Suppose, for example, that A_1 and A_2 have positive sums. Then in at least one row of the first 995 rows there are more black cells than white, which is a contradiction.

10. Since

$$\frac{a_1^2-a_2^2}{a_1+a_2}+\frac{a_2^2-a_3^2}{a_2+a_3}+\ldots+\frac{a_{n-1}^2-a_n^2}{a_{n-1}+a_n}+\frac{a_n^2-a_1^2}{a_n+a_1}=$$

$$=(a_1-a_2)+(a_2-a_3)+\ldots+(a_{n-1}-a_n)+(a_n-a_1)=0,$$

then

$$\frac{a_1^2}{a_1+a_2}+\frac{a_2^2}{a_2+a_3}+\ldots+\frac{a_{n-1}^2}{a_{n-1}+a_n}+\frac{a_n^2}{a_n+a_1}=$$

$$\frac{a_2^2}{a_1+a_2}+\frac{a_3^2}{a_2+a_3}+\ldots+\frac{a_n^2}{a_{n-1}+a_n}+\frac{a_1^2}{a_n+a_1}.$$

Using the following obvious inequality

$$\frac{a_i^2+a_j^2}{a_i+a_j}\geq\frac{1}{2}(a_i+a_j),$$

we get

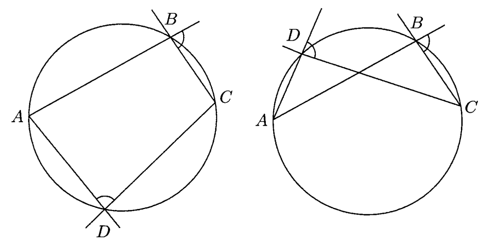
$$\frac{a_1^2}{a_1+a_2}+\frac{a_2^2}{a_2+a_3}+\ldots+\frac{a_{n-1}^2}{a_{n-1}+a_n}+\frac{a_n^2}{a_n+a_1}=$$

$$=\frac{1}{2}\left(\frac{a_1^2+a_2^2}{a_1+a_2}+\frac{a_2^2+a_3^2}{a_2+a_3}+\ldots+\frac{a_{n-1}^2+a_n^2}{a_{n-1}+a_n}+\frac{a_n^2+a_1^2}{a_n+a_1}\right)$$

$$\geq\frac{1}{2}(a_1+a_2+\ldots+a_n)=\frac{1}{2},$$

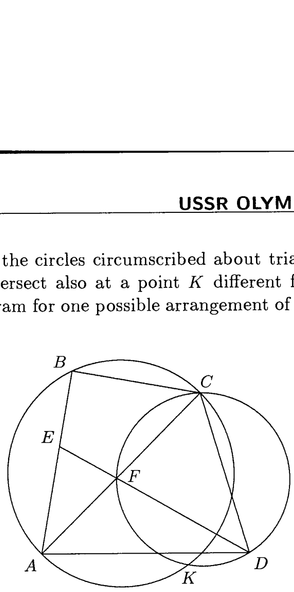
and the equality holds only for $a_1=a_2=\ldots=a_n=\frac{1}{n}$.

11. For two lines l and l', let us denote by $\angle(l,l')$ the least angle through which it is necessary to rotate l clockwise to obtain a line parallel to l'. It is well-known that, given three noncollinear points A, B, C, any point D, different from A and C, is concyclic with A, B and C if and only if $\angle(AB,BC)=\angle(AD,DC)$. Looking at the diagram below, note that in such formulation we make no distinction between the cases when D belongs to the arc ABC and the case when it belongs to the complementary arc.



We need such a form of this well-known theorem to avoid considering too many cases. Due to this, our proof will not depend on diagrams.

(a) Suppose that the circles circumscribed about triangles ABC and CDF intersect also at a point K different from C (see the next diagram for one possible arrangement of the circles).



All we need to prove is that the points B, D, E, K are concyclic. The statement is true if K coincides with B or D, so we may assume that $K \neq B$ and $K \neq D$. The points C, F, A are distinct points on a line and the points C, F, K are distinct points on a circle, yielding $K \neq A$. Thus K is different from A, B, C, D.

We know that

$$\angle(AC, CK) = \angle(AB, BK) \text{ and } \angle(FC, CK) = \angle(FD, DK).$$

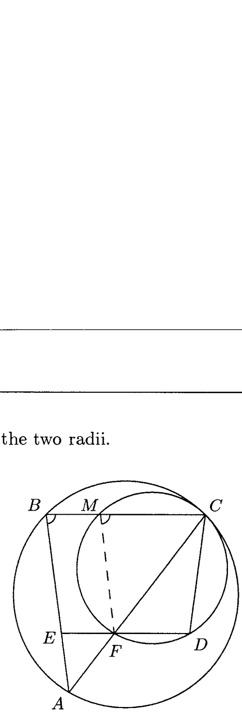
Hence

$$\angle(EB, BK) = \angle(AB, BK) = \angle(AC, CK) = \angle(FC, CK) = \angle(FD, DK) = \angle(ED, DK),$$

and therefore B, D, E, K are indeed concyclic.

(b) It remains only to consider the case when the circumscribed circles about ABC and CDF have only one common point C. In this case they are tangent internally (see diagram).

It is sufficient to prove that the points B, C, D, E are concyclic. Denote by M the point of intersection of the segment BC with the circle circumscribed about the triangle CDF. The triangles ABC and FMC are homothetic with centre of homothety C, and the coefficient of homothety equal to the ratio of the two radii.



It follows that $\angle ABC = \angle FMC$. The points B and D, and therefore the points M and D, lie on different sides of the diagonal AC, because $ABCD$ is convex. Since the quadrilateral $CDFM$ is cyclic, $\angle FMC + \angle FDC = 180°$. Hence $\angle ABC + \angle FDC = 180^0$ and the quadrilateral $BCDE$ is also cyclic.

12. The marked points divide the segment $[0, 1]$ into smaller segments, which we shall call intervals.

It is easy to verify that, if the points $9/23, 17/23, 19/23$ are marked, it is impossible for the grasshoppers to jump into the same interval in one move. Let us prove that no matter which points are marked they can always appear in the largest interval in two moves. It is sufficient to prove this only for the grasshopper sitting at 0: the same proof will be valid for the other grasshopper (or consider the mirror image of the interval with respect to $1/2$). Let $[\alpha, \alpha + s]$ be the largest interval. (If there are several intervals of length s choose an arbitrary one.) If $\alpha < s$, the grasshopper can jump into this interval in one move, jumping from 0 to 2α. If $\alpha \geq s$, consider the segment $[(\alpha - s)/2, (\alpha + s)/2]$ of length s. By the maximality of s this segment contains at least one marked point β. Jumping from 0 to 2β, the grasshopper will appear in the interval $[\alpha - s, \alpha + s]$. If it is not in the interval $[\alpha, \alpha + s]$, after a jump over the point α it will be.

13. Answer: $x = 584$.

It is clear that x is a natural number which is not greater than 1001, hence $x < 6!$ and thus the summands $[x/n!]$ for $n \geq 6$ can

be omitted. Each number $x < 6!$ can be represented in a unique way as $x = a \cdot 5! + b \cdot 4! + c \cdot 3! + d \cdot 2! + e \cdot 1!$, where a, b, c, d, e are non-negative integers such that $a \le 5$, $b \le 4$, $c \le 3$, $d \le 2$, $e \le 1$. Substituting this expression for x into the given equation we get another one

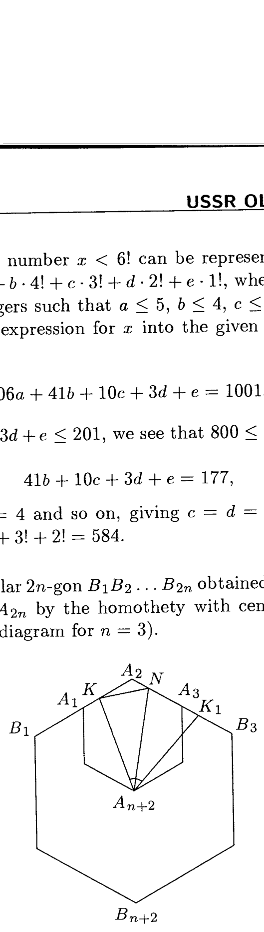
$$206a + 41b + 10c + 3d + e = 1001.$$

Since $41b + 10c + 3d + e \le 201$, we see that $800 \le 206a \le 1001$ and $a = 4$. Thus

$$41b + 10c + 3d + e = 177,$$

which implies $b = 4$ and so on, giving $c = d = 1$, $e = 0$. Thus, $x = 4 \cdot 5! + 4 \cdot 4! + 3! + 2! = 584$.

14. Consider the regular $2n$-gon $B_1B_2 \dots B_{2n}$ obtained from the regular $2n$-gon $A_1A_2 \dots A_{2n}$ by the homothety with centre A_2 and scale factor 2 (see the diagram for $n = 3$).



The point A_{n+2} is the centre of the polygon $B_1A_2B_3 \dots B_{2n}$. By rotating about the point A_{n+2} through angle π/n, the point K, lying on the side B_1A_2 of $B_1A_2B_3 \dots B_{2n}$, will be taken to a point K_1 on the side A_2B_3, and, thus, on the line NA_3. It follows from the equalities $K_1A_{n+2} = KA_{n+2}$ and $\angle K_1A_{n+2}K = \pi/n$ that the triangles $K_1A_{n+2}N$ and $KA_{n+2}N$ are congruent, since they have a common side $A_{n+2}N$ and

$$\begin{aligned} \angle K_1A_{n+2}N &= \angle K_1A_{n+2}K - \angle KA_{n+2}N \\ &= \pi/n - \pi/2n = \pi/2n = \angle KA_{n+2}N. \end{aligned}$$

Therefore $\angle KNA_{n+2} = \angle A_{n+2}NA_3$, and the statement is proved.

15. Answer: $2k$.

Let n be the number of vertices of the given polygon. Since the problem is actually dealing with a graph it is better not to distinguish between sides and diagonals, and we shall call them segments. Let us prove, first, that if the conditions of the problem are satisfied, then the number of segments of each colour does not exceed $n - 1$. Suppose that n segments are coloured the same colour, say green. We shall find a green closed broken line passing through vertices. To do this, we remove those vertices one by one from which only one green segment originates, along with the green segment (but not the other vertex of this segment). Initially the number of green segments and the number of vertices were equal, so they will be equal after each step of the procedure. We continue this process as long as possible.

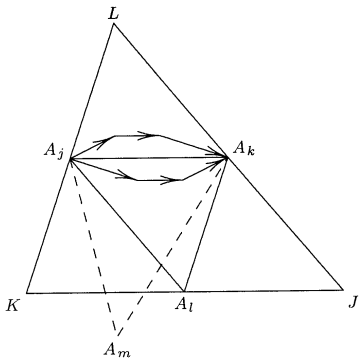
Of course the process terminates eventually. But what do we have at the end? It is impossible to remove all segments and all vertices. Indeed, suppose that a vertex A and a green segment AB were the last to be deleted. Then the vertex B must been removed earlier. But this is not possible since at the moment of its deletion there would have been at least two green segments originating from B. Thus we cannot delete all vertices and our process terminates when we discover that at least two green segments originate from all remaining vertices. But in this situation it is obvious that a green closed loop exists.

In total an n-gon has $(n-1)n/2$ segments (sides and diagonals), thus we obtain the inequality $(n-1)n/2 \le k(n-1)$, and hence $n \le 2k$. It remains only to show that for $n = 2k$ the required colouring exists. Let us take a regular $2k$-gon $A_1A_2 \dots A_{2k}$ and colour the broken line $A_1A_{2k}A_2A_{2k-1} \dots A_kA_{k+1}$ with the first colour. We colour the broken line obtained by rotating this broken line through angle $i\pi/k$ with the i-th colour. Because the rotation is only a convenient way of describing the colouring a similar method works for non-regular polygons.

16. Let us draw all n vectors $\vec{a}_1, \dots, \vec{a}_n$, starting from A_0. Set $\vec{a}_{i_1} = \vec{a}_1$ and let $\vec{a}_{i_s}$ be the sth vector that we get by rotating $\vec{a}_1$ clockwise. Since the sum of the vectors is zero, the angle between two successive vectors $\vec{a}_{i_s}$ and $\vec{a}_{i_{s+1}}$ is less than 180°. Therefore the permutation $\vec{a}_{i_1}, \vec{a}_{i_2}, \dots, \vec{a}_{i_n}$ gives us a convex polygon $A_1A_2 \dots A_n$, where $A_n = A_0$. For convenience of notation we assume that the vectors $\vec{a}_1, \dots, \vec{a}_n$ were arranged in this order at the very beginning. It will also be convenient to let $A_{n+m} = A_m$.

Consider all triangles, whose vertices are also vertices of the polygon, and let $A_jA_kA_l$, where $j < k < l$ be the one among them with the largest area. Through each vertex of the triangle $A_jA_kA_l$ draw a line parallel to the opposite side (see diagram).

These lines intersect at points J, K, L as shown. From the maximality of the area of $A_jA_kA_l$ it follows that all vertices of the polygon lie inside the triangle JKL. Otherwise we could find a triangle with area greater than that of $A_jA_kA_l$. (In the diagram the area of the dotted triangle $A_jA_kA_m$ is greater than the area of $A_jA_kA_l$.)



Since the n-gon $A_1A_2 \dots A_n$ is convex, the broken line $A_jA_{j+1} \cdots A_k$ lies inside the triangle A_jLA_k. Let us now reverse the order of the vectors

$$\vec{a}_{j+1} = \overrightarrow{A_jA_{j+1}},\ \vec{a}_{j+2} = \overrightarrow{A_{j+1}A_{j+2}},\ \dots,\ \vec{a}_k = \overrightarrow{A_{k-1}A_k}$$

in the sequence $\vec{a}_1, \dots, \vec{a}_n$. Then the new broken line corresponding to the new permutation of vectors will be

$$A_1 \dots A_{j-1}A'_jA'_{j+1} \cdots A'_kA_{k+1} \cdots A_n$$

where the new segment $A'_jA'_{j+1} \cdots, A'_k$ is symmetric to the old segment $A_jA_{j+1} \dots A_k$ with respect to the midpoint of A_jA_k. The segment $A'_jA'_{j+1} \cdots A'_k$ will now be inside $A_jA_kA_l$. We repeat the same operation with the segments $A_kA_{k+1} \dots A_l$ and $A_lA_{l+1} \cdots A_j$

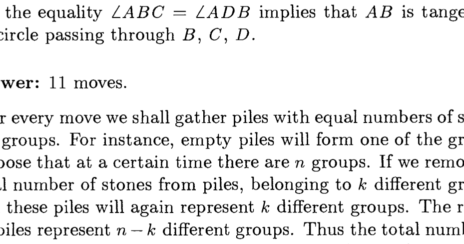
after which all points of the broken line $A'_1, A'_2, \dots, A'_n$ will lie inside $A_jA_kA_l$. Again we assume that the vectors $\vec{a}_1, \dots, \vec{a}_n$ were arranged in this order at the very beginning as we will not need convexity any more.

One of the angles of $A_jA_kA_l$, say $\angle A_lA_jA_k$, is less than or equal to $60°$. It is clear now that the following cyclic permutation

$$\vec{a}_{j+1}, \dots, \vec{a}_n, \vec{a}_1, \dots, \vec{a}_j$$

of the vectors satisfies the required condition.

17. The two given circles are homothetic with A as the centre of similitude. Hence the angles $\angle ABC$ and $\angle ADB$ that subtend homothetic arcs CE and DB are equal.



Turning our attention to the circle passing through B, C, D we note that the equality $\angle ABC = \angle ADB$ implies that AB is tangent to the circle passing through B, C, D.

18. Answer: 11 moves.

After every move we shall gather piles with equal numbers of stones into groups. For instance, empty piles will form one of the groups. Suppose that at a certain time there are n groups. If we remove an equal number of stones from piles, belonging to k different groups, then these piles will again represent k different groups. The rest of the piles represent $n - k$ different groups. Thus the total number of groups after the move will not be less than $\max(k, n-k)$. Hence the number of different groups decreases no faster than the sequence

$$995, 498, 249, 125, 63, 32, 16, 8, 4, 2, 1$$

so we have to make at least 11 moves.

Let m_n be the move in which we remove n stones from each pile containing n stones or more. It is easy to check that the 11 moves

$$m_{995}, m_{498}, m_{249}, m_{125}, m_{63}, m_{32}, m_{16}, m_8, m_4, m_2, m_1$$

remove all stones.

19. Note that if $x_1 = 1$, then $f(x_1) = a+b+c = 1$. To proceed further, we shall prove that for any positive x, y the following inequality holds

$$f(x) \cdot f(y) \geq (f(\sqrt{xy}))^2. \tag{1}$$

Indeed, if we denote $\sqrt{xy}$ by z, then

$$f(x) \cdot f(y) - (f(z))^2 = a^2(x^2y^2 - z^4) + b^2(xy - z^2) + c^2(1-1)+$$
$$ab(x^2y + xy^2 - 2z^3) + ac(x^2 + y^2 - 2z^2) + bc(x + y - 2z) =$$
$$= ab(\sqrt{x^2y} - \sqrt{xy^2})^2 + ac(x-y)^2 + bc(\sqrt{x} - \sqrt{y})^2 \geq 0.$$

Now for all n equal to a power of 2 we will prove by induction that for all positive $x_1, \ldots, x_n$

$$f(x_1) \cdot \ldots \cdot f(x_n) \geq \left(f\left(\sqrt[n]{x_1 \cdot \ldots \cdot x_n}\right)\right)^n.$$

Suppose, this is true for $n = 2^k$. Then using the inductive hypothesis and (1) we obtain

$$\begin{aligned} & f(x_1) \cdot \ldots \cdot f(x_{2^{k+1}}) \\ = \; & (f(x_1) \cdot \ldots \cdot f(x_{2^k})) \cdot (f(x_{2^k+1}) \cdot \ldots \cdot f(x_{2^{k+1}})) \\ \geq \; & \left(f\left(\sqrt[2^k]{x_1 \cdot \ldots \cdot x_{2^k}}\right) \cdot f\left(\sqrt[2^k]{x_{2^k+1} \cdot \ldots \cdot x_{2^{k+1}}}\right)\right)^{2^k} \\ \geq \; & \left(f\left(\sqrt[2^{k+1}]{x_1 \cdot \ldots \cdot x_{2^{k+1}}}\right)\right)^{2^{k+1}}, \end{aligned}$$

and so the statement is also true for $n = 2^{k+1}$.

Suppose now that n is arbitrary, and $x_1 \cdot \ldots \cdot x_n = 1$. Let k be the positive integer such that $2^{k-1} < n \leq 2^k$. Let us add, if necessary, $x_{n+1} = x_{n+2} = \ldots = x_{2^k} = 1$. Since $f(x_{n+1}) = f(x_{n+2}) = \ldots = f(x_{2^k}) = 1$, we may write

$$f(x_1) \cdot \ldots \cdot f(x_n) = f(x_1) \cdot \ldots \cdot f(x_{2^k}) \geq$$
$$\left(f\left(\sqrt[2^k]{x_1 \cdot \ldots \cdot x_{2^k}}\right)\right)^{2^k} = 1.$$

20. Answer: yes, it is possible.

Let us introduce a rectangular coordinate system in such a way that each coordinate of each vertex of the large cube is either 0 or 100. Any straight segment of the grid of length 100 we shall call a diameter. Let us mark some of the vertices of the grid so that on each diameter there is exactly one marked vertex. For instance, mark all vertices for which the sum of the coordinates is equal to a multiple of 101. For any vertex A, which is not marked, we construct a frame $F(A)$ with common vertex A as follows. For every diameter of the grid passing through A we include in $F(A)$ the unit segment originating from A, which goes in the direction of the only marked point of this diameter. It is clear that the frames, so constructed, have no common edges. Moreover, each unit edge AB belongs to a certain frame. Indeed, at least one of the points, A or B, is not marked. If, say, A is marked, then the edge AB belongs to $F(B)$. If neither is marked, then one of the rays AB or BA contains a marked vertex of the grid. In this case, the edge AB belongs to $F(A)$ or $F(B)$, respectively.

21. Answer: for $n > 1$.

Indeed,

$$3^{2n+1} - 2^{2n+1} - 6^n = (3^n - 2^n)(3^{n+1} + 2^{n+1}).$$

For $n > 1$ this number is composite since $3^n - 2^n > 1$. For $n = 1$ it is 13, which is prime.

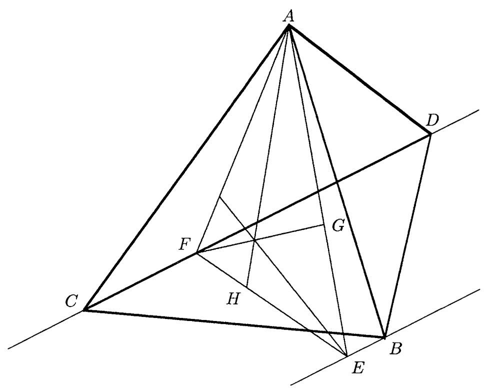
22. Let h be the length of the altitude AH of the tetrahedron $ABCD$ drawn from the vertex A, and d be the distance between AB and CD. Let us draw a line l through the vertex B, parallel to CD, and consider also a plane passing through the vertex A perpendicular to CD.

This plane contains the altitude AH, and it intersects l and CD at the points E and F. Then the altitudes AH and FG of the triangle AEF are equal to h and d, respectively.

Since the third altitude of the triangle AEF is equal to one of the altitudes of the tetrahedron $ABCD$, it cannot be less than h.

Therefore $AF \leq FE$ and

$$\frac{h}{d} = \frac{AH}{FG} = \frac{AE}{FE} < \frac{AF + FE}{FE} \leq 2;$$



which is what was to be proved.

23. Answer: yes, it is possible.

It will be useful during the solution of this problem to keep in mind Vieta's formulas for a cubic equation; namely, if

$$x^3 + ax^2 + bx + c = (x - x_1)(x - x_2)(x - x_3),$$

then

$$\begin{aligned} a &= -(x_1 + x_2 + x_3), \\ b &= x_1x_2 + x_1x_3 + x_2x_3, \\ c &= -x_1x_2x_3. \end{aligned}$$

One strategy is as follows. At the beginning the first player chooses 0. If the second player makes it the constant term of the equation, the equation will be

$$x^3 + \ldots x^2 + \ldots x = 0,$$

and the first player chooses successively 2 and -3 to obtain

$$x(x-1)(x+3) = 0 \quad \text{or} \quad x(x-1)(x-2) = 0.$$

If the second player puts 0 as the coefficient of x^2, the equation will be $x^3 + bx + c = 0$, with b and c not fixed yet. The first player

chooses the number $-(3 \cdot 4 \cdot 5)^2$ and then, depending on the move of the second player, either $c = 0$ or $b = 3^2 \cdot 4^2 - 3^2 \cdot 5^2 - 4^2 \cdot 5^2$. This will result in the equations $x(x - 3 \cdot 4 \cdot 5)(x + 3 \cdot 4 \cdot 5) = 0$ or $(x + 3^2)(x + 4^2)(x - 5^2) = 0$.

If after the move of the second player the equation is $x^3 + ax^2 + c = 0$, the first player chooses $6^2 \cdot 7^3$ and then either $a = -49$ or $c = -6^8 \cdot 7^6$ to get the equations

$$(x + 2 \cdot 7)(x - 3 \cdot 7)(x - 6 \cdot 7) = 0$$

or

$$(x - 2 \cdot 6^2 \cdot 7^2)(x + 3 \cdot 6^2 \cdot 7^2)(x + 6 \cdot 6^2 \cdot 7^2) = 0.$$

24. We shall prove by induction on the even number n, that if n coins are given and if it is known how many coins among them are counterfeit, then it is possible to determine all counterfeit coins in $\lceil 3n/4 \rceil$ weighings.

For $n = 2$ the statement is true since everything can be determined in one weighing. Suppose that $n \geq 4$. Let us compare two arbitrary coins. If they have different weights, we can classify them and since

$$\left\lceil \frac{3(n-2)}{4} \right\rceil + 1 \leq \left\lceil \frac{3n}{4} \right\rceil,$$

the problem reduces to the same problem for $n - 2$ coins.

Suppose that the compared coins have equal weights. Then we shall compare this pair of coins with another pair. If the weights differ, then we shall compare the two coins of the second pair and all four coins after that will be classified. Since

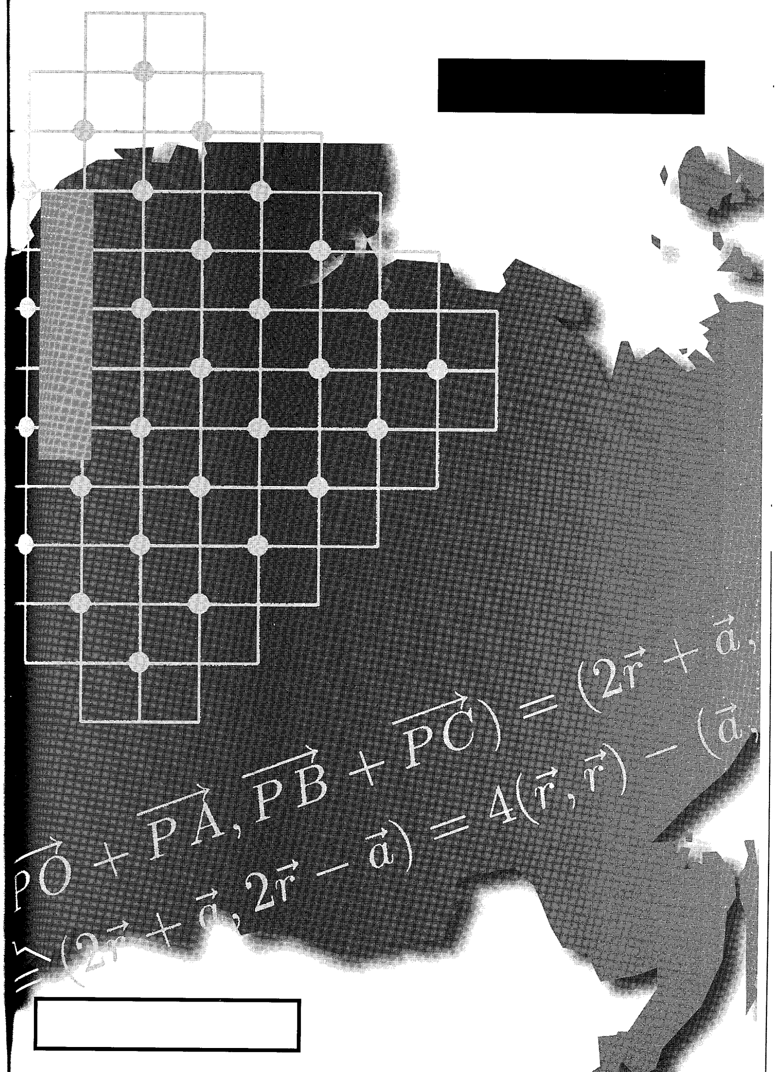
$$\left\lceil \frac{3(n-4)}{4} \right\rceil + 3 \leq \left\lceil \frac{3n}{4} \right\rceil,$$

the problem reduces to the problem for $n - 4$ coins. If the two pairs have equal weights, we compare these four coins with another four. If these two sets of coins differ in weight, then the first four can be classified and the problem again reduces to $n - 4$ coins. If the two sets have equal weights, we compare these eight coins with another eight and so on.

If at some stage of this procedure a group of 2^m coins differs from another group of 2^m coins, then the coins of the first group (which are identical) can be classified. Since

$$\left\lceil \frac{3(n-2^m)}{4} \right\rceil + (m-1) \leq \left\lceil \frac{3n}{4} \right\rceil,$$

the problem reduces to the problem for $n - 2^m$ coins. Finally, if there are not enough coins to form another group of 2^m coins, then $2^m > n/2$ and the 2^m coins can be classified by choosing $n - 2^m$ of them and comparing them with the remaining $n - 2^m$ coins.



USSR OLYMPIAD 1991

9 FORM

First day

1. Find all integer solutions of the system

$$\begin{cases} xz - 2yt = 3, \\ xt + yz = 1. \end{cases}$$

(Yu Nesterenko, Moscow)

2. Given n numbers on a blackboard, it is permitted to erase any pair of them, say a and b, and to write down the number $\frac{a+b}{4}$ instead of them. When this procedure is repeated $n-1$ times, only one number remains on the blackboard. Suppose that all n numbers at the beginning were equal to 1. Prove that the last number will not be less than $\frac{1}{n}$.

(B Berlov)

3. Four straight lines are given in a plane such that each two of them intersect but no three of them have a point in common. Each line is divided into two rays and two segments by three points of intersection. The total number of segments is eight. Is it possible that the lengths of these segments are equal to

(a) $1, 2, 3, 4, 5, 6, 7, 8$?

(b) pairwise distinct natural numbers?

(A Berzinsh, Riga)

4. A lottery ticket takes the form of a card which has 50 empty cells in a row. A participant, on each of his tickets, writes down in these empty cells the numbers $1, 2, \ldots, 50$ without a repetition. The organizer of a lottery has his own ticket, with the numbers written on it according to the same rule. A ticket wins if at least in one cell of it the number coincides with the number in the corresponding cell of the organizer's ticket. What is the least possible number of tickets, which a participant has to fill in, to guarantee that at least one of his ticket wins?

(A Berzinsh, Riga)

9 FORM

Second day

5. (a) Find a pair of integers x, y such that $xy + x$ and $xy + y$ are squares of distinct natural numbers.

(b) Is it possible to find such numbers x, y, both between 988 and 1991?

(A Andzans, Riga)

6. A rectangle $ABCD$ is given. Points K, L, M, N, distinct from vertices, are chosen on the sides AB, BC, CD, DA, respectively, so that $KL \parallel MN$ and $KM \perp NL$. Prove that the point of intersection of KM and LN lies on the diagonal BD of the rectangle.

(D Tereshin, Moscow)

7. A judge has a plan for crime investigation. Questioning the witness, he is allowed to ask only questions which are to be answered "yes" or "no". The judge is sure that all answers will be true, and he calculates that under this condition no more than 91 questions will be needed to find out the truth. (The questions may depend on the answers to the previous ones.) Show that the judge can investigate this crime, using no more than 105 questions, if it is known that one answer may be a lie.

(A Andzans, Riga; I Solov'ev, V Slitinski, Moscow)

8. In the cells of a square 5×5 board one minus and 24 pluses are written down. In a move it is permitted to choose a square on the board, consisting of more than one cell, and change all signs in this square simultaneously to the opposite ones. Find all positions of the minus such that it is possible to obtain all pluses on the board after several moves.

(A Grintsyavichus, Vilnjus)

10 FORM

First day

9. Prove that the inequality

$$\frac{(a+b+c)^2}{3} \geq a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}$$

holds for all nonnegative numbers a, b, c.

(D Tereshin, Moscow)

10. Does there exist a triangle in which the lengths of

(a) two sides,

(b) three sides

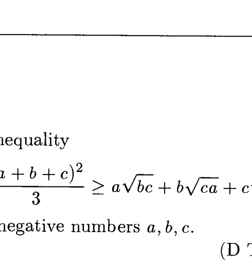
are integer multiples of the lengths of the corresponding medians drawn to these sides?

(N Agahanov, Moscow)

11. Several consecutive natural numbers $1, 2, 3, \ldots, n$ (more than two) are written down on a blackboard. In one move it is permitted to erase any pair of numbers, say p and q, and to write down numbers $p + q$ and $|p - q|$ instead of them. A student in several moves could make all numbers on the blackboard equal to k. Find all possible values of k.

(B Merkulov, Moscow)

12. The figure shown below



is cut along the lines into a number of polygons (not necessarily convex), none of which contains a 2×2 square. Find the smallest possible number of these polygons.

(A Andzans, Riga)

10 FORM

Second day

13. An acute-angled triangle ABC is inscribed in a circle with centre D. The circle drawn through the points A, B, D, intersects the sides AC and BC at points M and N, respectively. Prove that the radii of the circles circumscribed about the triangles ABD and MNC are equal.

(B Chinik, Kishinev)

14. A new polygon can be obtained from an old one by the following procedure. The old polygon is divided into two polygons by a straight cut, then one of the two obtained polygons is turned over and joined with the remaining polygon by joining the sides, created by the cut. This procedure is allowed only if the two polygons do not overlap after being joined. Is it possible to transform a square into a triangle, by applying this procedure several times?

(I Voronovich, Minsk)

15. A $k \times l$ minor of an $n \times n$ table consists of all cells which lie on the intersection of any k rows with any l columns. The number $k + l$ is called the semiperimeter of this minor. It is known that several minors of semiperimeters not less than n each, jointly cover the main diagonal of the table. Prove that these minors jointly cover at least half of all cells.

(D Flaas, Novosibirsk)

16. Let $a_1, a_2, \ldots, a_{100}, b_1, b_2, \ldots, b_{100}$ be distinct real numbers. A table of 100×100 numbers is filled so that the number $a_i + b_j$ is written down on the intersection of the ith row with the jth column. Given that the product of all numbers in each column is equal to 1, prove that the product of all numbers in each row is equal to -1.

(D Fomin, St. Petersburg)

11 FORM

First day

17. In a sequence of positive integers the number a_{n+1} is obtained from a_n by the following rule. If the last decimal digit of a_n does not exceed 5, then this last digit is removed and the remaining sequence of digits is taken as the decimal representation of a_{n+1}; if this last digit was the only digit of a_n the process terminates. Otherwise, $a_{n+1} = 9a_n$. Can a_0 be chosen so that this process is infinite?

(A Azamov, Tashkent; S Konyagin, Moscow)

18. Suppose the real numbers α and β satisfy the equations

$$\alpha^3 - 3\alpha^2 + 5\alpha = 1, \qquad \beta^3 - 3\beta^2 + 5\beta = 5.$$

Find $\alpha + \beta$.

(B Kukushkin, Moscow)

19. Points A, B, C, D, E are situated on a sphere, so that the chords AB and CD intersect at a point F, and the points A, C, F are equidistant from the point E. Prove that BD and EF are perpendicular.

(B Chinik, Kishinev; I Sergeev, Moscow)

20. Does there exist a set of vectors in a plane consisting of:

(a) 4 noncollinear vectors such that the sum of each 2 of those vectors is perpendicular to the sum of the other 2?

(b) 91 nonzero vectors with the sum of any 19 of them perpendicular to the sum of all others?

(D Fomin, St. Petersburg)

11 FORM

Second day

21. On the sides AB and AD of a square $ABCD$ points K, N are chosen, respectively, so that $AK \cdot AN = 2BK \cdot DN$. The lines CK and CN intersect the diagonal BD at points L and M. Prove that the points K, L, M, N, A are concyclic.

(D Tereshin, Moscow)

22. There were 100 mutually conflicting countries on the planet Xenon. For maintaining peace several military alliances were established. No one alliance contains more than 50 countries, and each pair of countries belongs to at least one alliance. What is the smallest possible number of alliances? What would this number be under the additional condition that no pair of alliances jointly contains more than 80 countries?

(D Flaas, Novosibirsk)

23. Given $2n$ distinct real numbers $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$, a table of 100×100 numbers is filled so that the number $a_i + b_j$ is written down on the intersection of the ith row with the jth column. Given that in each column the products of all numbers are equal, prove that in each row the products of all numbers are also equal.

(D Fomin, St Petersburg)

24. The numbers $x_1, x_2, \ldots, x_{1991}$ satisfy the equation

$$|x_1 - x_2| + |x_2 - x_3| + \ldots + |x_{1990} - x_{1991}| = 1991.$$

What is the greatest possible value of the expression

$$|y_1 - y_2| + |y_2 - y_3| + \ldots + |y_{1990} - y_{1991}|,$$

where $y_k = \frac{1}{k}(x_1 + x_2 + \ldots + x_k)$?

(A Kachurovskii, Novosibirsk)

SOLUTIONS

1. The system is equivalent to the following one equation in complex numbers

$$(x + i\sqrt{2}y)(z + i\sqrt{2}t) = 3 + i\sqrt{2}.$$

Comparing moduli we get

$$(x^2 + 2y^2)(z^2 + 2t^2) = 11,$$

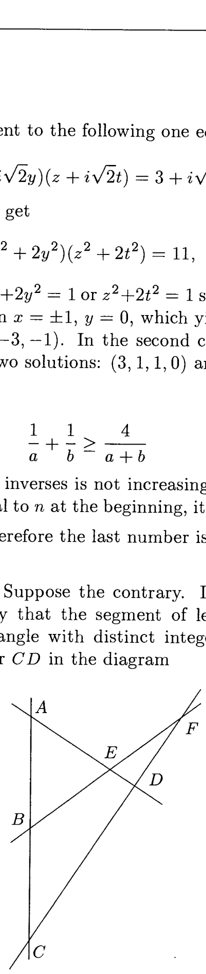
and therefore either $x^2+2y^2 = 1$ or $z^2+2t^2 = 1$ since 11 is prime. In the first case we obtain $x = \pm 1$, $y = 0$, which yields two solutions: $(1, 0, 3, 1)$ and $(-1, 0, -3, -1)$. In the second case $z = \pm 1$, $t = 0$ and we again obtain two solutions: $(3, 1, 1, 0)$ and $(-3, -1, -1, 0)$.

2. The inequality

$$\frac{1}{a} + \frac{1}{b} \geq \frac{4}{a+b}$$

shows that the sum of inverses is not increasing during the procedure. Since it was equal to n at the beginning, it will not be greater than n at the end. Therefore the last number is not less than $\frac{1}{n}$.

3. (a) It is impossible. Suppose the contrary. It follows from the triangle inequality that the segment of length 1 cannot be a side of any triangle with distinct integers as sidelengths. Hence only BC or CD in the diagram



can have length equal to 1. Suppose that $BC = 1$. Since both BF and CF have integer lengths, they are necessarily equal and the triangle BFC is isosceles. Since BC is the shortest

side of BFC the angle $\angle F$ is acute. Applying the Cosine theorem to BFC we obtain

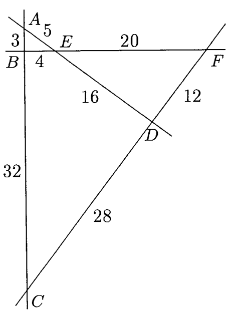
$$\cos F = 1 - \frac{1}{2BF^2}.$$

Applying the same theorem to the triangle EFD we obtain

$$ED^2 = EF^2 + FD^2 - 2EF \cdot FD + \frac{EF \cdot FD}{BF^2}.$$

Since $EF < BF$ and $FD < CF = BF$ the last summand of the right-hand-side is not an integer and the equation above cannot hold. This is a contradiction.

(b) It is possible. An example is shown in the following diagram.



The construction starts with the Pythagorean triangle ABE.

4. Answer: 26 tickets.

To guarantee a win, 26 tickets can be filled as follows:

1	2	3	...	25	26	27	...	50
2	3	4	...	26	1	27	...	50
3	4	5	...	1	2	27	...	50
...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
25	26	1	...	23	24	27	...	50
26	1	2	...	24	25	27	...	50

In fact, one of the numbers $1, 2, 3, \ldots, 26$ must be in the first 26 cells of the organizer's ticket (since there are only 24 other cells).

This number will coincide with the corresponding number in one of the 26 tickets shown, and this ticket will win.

To show that 25 tickets do not guarantee a win, let us take 25 arbitrary tickets filled in by some participant of the lottery (see the diagram)

a_1	a_2	a_3	...	a_{50}
b_1	b_2	b_3	...	b_{50}
...	...	...	...	...
c_1	c_2	c_3	...	c_{50}
			...	

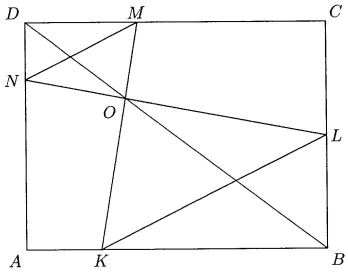
and prove that the organizer's ticket shown in the last row could be filled in in such a way, that all 25 tickets above it lose. The algorithm is as follows.

It is clear that it is possible to write down 1 in a cell of the organizer's ticket so that it will differ from all 25 numbers written in the corresponding cells of the 25 participant's tickets. Suppose that $1, 2, \ldots, a-1$ are already written down in the organiser's ticket as required. Let us try to write down the number a. We have at least 25 cells in the organizer's ticket where a could be written, but they can be occupied with some other numbers $x_1, x_2, \ldots, x_{25}$ which were written down earlier. Choose an arbitrary empty cell z of the organizer's ticket. It is suitable at least for 25 numbers, but not for a. Therefore it is suitable for one of the numbers $x_1, x_2, \ldots, x_{25}$. Therefore we can remove one of these numbers to z and clear the cell suitable for a. Write down a in this cell and continue.

5. (a) For example, $x = 1$, $y = 8$.

(b) Suppose that $y > x$. Then $x^2 < xy + x < xy + y$, and $xy + x = (x+i)^2$, $xy + y = (x+j)^2$, where $i \geq 1$ and $j \geq 2$. The difference of the two squares $y - x = (xy+y) - (xy+x) = (x+j)^2 - (x+i)^2 = 2(j-i)x + (j^2 - i^2) > 2x$. Hence $y > 3x$, and it is impossible for x and y to be both between 988 and 1991.

6. Let us denote the point of intersection of KM and LN by O, and draw the lines BO and DO.



Solution 1. Since $\angle NOM = \angle NDM = 90°$, the points D and O are on the circle with diameter NM and $\angle NOD = \angle NMD$. Similarly $\angle LOB = \angle LKB$. But $\angle LKB = \angle NMD$ as NM is parallel to KL. Therefore $\angle NOD = \angle LOB$ and the points D, O, B are collinear.

Solution 2. This solution does not rely on the condition that $KM \perp NL$. Since NM is parallel to KL the triangle NDM is homothetic to the triangle KLB and therefore NDM is similar to KLB with coefficient of similarity which we will denote by α. For the same reason the triangle NOM is also similar to triangle LOM, with the same coefficient of similarity α. Therefore

$$\frac{LB}{ND} = \frac{LO}{NO} = \alpha.$$

Moreover $\angle DNO = \angle BLO$, and therefore triangle NOD is similar to triangle LOB. Hence $\angle LOB = \angle NOD$, and BOD is a straight line.

7. Let us denote the judge's initial plan by S. If one of the witness' answers may be a lie the plan can be modified as follows. The judge should divide all 91 questions of S into 13 series consisting of $13, 12, 11, 10, \ldots, 3, 2, 1$ questions, respectively. He should also prepare an additional checking question:

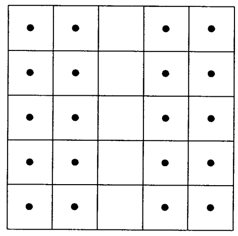
"Did you lie answering questions of the last series?"

The judge should start with the questions of the first series and then ask the checking question. If the answer to the checking question

is "No", there were no wrong answers, and he can proceed with the second series ending it again with the checking question. If there were no "Yes" answers to the checking questions the judge can be certain that all questions of S were answered truthfully. In this case the judge uses $91 + 13 = 104$ questions.

If the judge receives the answer "Yes" to the checking question after the ith series, the judge should repeat this series and further follow S without asking any additional questions. In this case the judge needs i checking questions and $14 - i$ repetitions of the questions of the ith series. In all, the judge needs $91 + i + (14 - i) = 105$ questions.

8. In the diagram

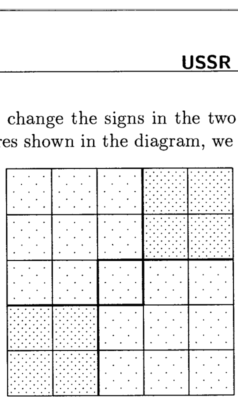


each square, consisting of more than one cell, contains an even number of marked cells. Thus, if the initial minus was in a marked cell, then after any move there will be an odd number of minuses in the marked cells and hence at least one.

We can also mark all cells in the two upper and two lower rows. Then the same reasoning proves that we cannot get rid of a minus in these marked cells also. Therefore no one position of the minus allows us to get rid of it, except maybe the position in the centre of the square.

If the minus is in the central cell, all pluses on the board can be obtained easily in several ways.

For example, if we change the signs in the two 2×2 squares and the two 3×3 squares shown in the diagram, we obtain all minuses.



And it remains to change the signs in the whole 5×5 square to get all pluses.

9. Using the Arithmetic mean-geometric mean inequality, we obtain

$$(a+b+c)^2 = (a^2+b^2+c^2+ab+bc+ca)+\frac{1}{2}(2ab+2bc+2ca) =$$

$$(a^2+bc)+(b^2+ca)+(c^2+ab)+\frac{1}{2}((ab+ca)+(bc+ab)+(ca+bc)) \geq$$

$$\geq 2a\sqrt{bc}+2b\sqrt{ca}+2c\sqrt{ab}+\frac{1}{2}(2a\sqrt{bc}+2b\sqrt{ca}+2c\sqrt{ab}) =$$

$$= 3a\sqrt{bc}+3b\sqrt{ca}+3c\sqrt{ab}.$$

10. (a) **Answer:** such a triangle exists.

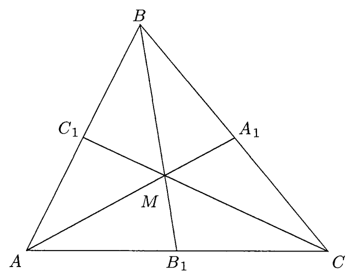
One example is the isosceles triangle with sides 2,2 and $\sqrt{6}$. Another example can be constructed using the following idea. In any right-angled triangle the median drawn to the hypotenuse is equal to half of the length of this hypotenuse. Hence to construct an example it is sufficient to choose any right-angled triangle in which the median drawn to the leg equals this leg. The reader may work out the details.

(b) **Answer:** No, it does not exist.

Suppose on the contrary that the triangle ABC satisfies all the required conditions. Let AA_1, BB_1, CC_1 be the medians of ABC drawn to the sides BC, AC, AB, respectively. Denote

$$\frac{BC}{AA_1} = m, \quad \frac{AC}{BB_1} = n, \quad \frac{AB}{CC_1} = k,$$

where m, n, k are integers. If $n \geq 2$, the point B is inside or on the circumference of the circular disc with the diameter AC. Thus $\angle ABC \geq \pi/2$ and therefore at least two of the three numbers m, n, k are equal to 1, say $m = k = 1$. This means that $AB = CC_1$ and $BC = AA_1$.



Let us assume that $AB \leq BC$. This implies that $\angle AB_1B \leq \pi/2$ and hence AM$\leq MC$, where M is the centroid of ABC. But $AM = \frac{2}{3}AA_1$ and $MC = \frac{2}{3}CC_1$. Thus $AA_1 \leq CC_1$ and $BC \leq AB$. Therefore $AB = BC$, and the triangle ABC is isosceles. Suppose now that $n \geq 2$. In this case $\angle ABC \geq \pi/2$ and $AA_1 > AB = BC$, which is a contradiction.

There remains only one possibility: $m = n = k = 1$. But in this case the triangle ABC is equilateral, with corresponding ratios equal to $\sqrt{3}/2$. These ratios are not integers so this possibility must also be rejected.

11. **Answer:** 2^s for any s satisfying the inequality $2^s \geq n$.

After any move there will only be nonnegative integers on the blackboard. If the sum and the difference of two integers are both divisible by an odd number d, then these numbers themselves are also divisible by d. Since 1 was among the initial numbers, k has no odd prime divisors. Thus $k = 2^s$. Since at any move the maximum of the numbers on the blackboard does not decrease, $k = 2^s \geq$n.

To prove that each number $k = 2^s \geq n$ can be obtained, we shall use induction. Firstly, let us note that if at some stage zero appears on the blackboard, then each number can be doubled in two moves:

$$(0, a) \to (a, a) \to (0, 2a).$$

Thus if the following zeros and powers of 2 are on the blackboard:

$$0,\ldots,0,2^{k_1},\ldots,2^{k_m},$$

with at least one zero, then after several moves it is possible to obtain

$$0,2^k,\ldots,2^k, \qquad k=\max(k_1,\ldots,k_m).$$

Lemma. There is a sequence of moves that transforms the numbers $1,2,\ldots,n$, where $n\geq 3$, into the numbers $0,2^{s+1},\ldots,2^{s+1}$, where s is the greatest integer satisfying the inequality $2^s<n$.

Proof. It is an easy exercise to verify this lemma for $3\leq n\leq 6$. For example,

$$1,2,3,4,5\to 1,2,2,4,8\to 0,1,4,4,8\to 0,8,8,8,8.$$

We then assume that $n>6$. Suppose that for $n'<n$ the statement of the lemma is true. Let us represent n in the form $n=2^s+b$, where $0<b\leq 2^s$. If $b=2^s$, then $n=2^{s+1}$ and the inductive step is trivial since by the induction hypothesis $1,2,\ldots,n-1$ can be transformed into $0,2^{s+1},\ldots,2^{s+1}$. Suppose that $0<b<2^s$ (note, that now $n=7$ or $n>8$) and divide $1,2,\ldots,n$ into four groups:

(a) $1,2,\ldots,2^s-b-1$;
(b) 2^s;
(c) $2^s-1,\ldots,2^s-b$;
(d) $2^s+1,\ldots,2^s+b$.

After b moves involving the pairs $(2^s+i,2^s-i)$ we get the following four groups:

(a) $1,2,\ldots,2^s-b-1$;
(b) 2^s;
(c) $2,4,6,\ldots,2b$;
(d) $2^{s+1},2^{s+1},\ldots,2^{s+1}$.

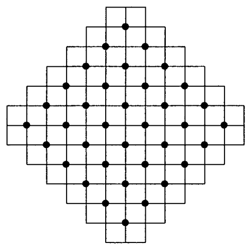
Since $b=3$ for $n=7$ and $(2^s-b-1)+b=2^s-1\geq 7$ for $n>8$, in the first or third group there are more than two numbers, so the induction hypothesis can be applied to this group giving at least one zero. For the other group either we can also apply the induction hypothesis or this group contains only powers of 2. In both cases we get only zeros and powers of 2 on the blackboard. This proves the lemma.

Now the solution to the problem can be easily obtained.

Suppose that the numbers $2^m,2^m,\ldots,2^m$ are to be obtained at the blackboard and $2^m\geq n$. First, using the lemma, we obtain numbers $0,2^{s+1},\ldots,2^{s+1}$. Then by doubling, if necessary, we can obtain $0,2^m,\ldots,2^m$ and finally $2^m,2^m,\ldots,2^m$.

12. Answer: 12 polygons.

Suppose that the figure is cut into n polygons as required. Let us consider one of the 36 points marked in the diagram.



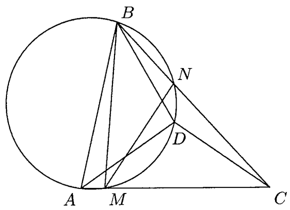
The 2×2 square centred on the marked point is not contained in any one of the polygons. Hence there is a cut passing through the marked point in the centre of this square.

We shall call any side of a 1×1 square which is not on the boundary of the figure an edge. For every marked point, at least two edges with ends in the marked point belong to the cut. Therefore we have no more than $144-2\cdot 36=72$ edges, not belonging to the cut.

Let us now cut the figure along all the edges which do not belong to the cut. We obtain 84 1×1 squares. Since any cut along an edge can increase the total number of polygons by no more than 1 we have $n+72\geq 84$ or $n\geq 12$.

It is easy to cut the figure into exactly 12 polygons of the required type: cut all horizontal edges and you will obtain 12 strips of width 1, which obviously satisfy the conditions.

13. The triangles BNM and MNC have the side MN in common. Hence it is sufficient to prove that $\angle MBN = \angle MCN$.



Since D is the centre of the circle circumscribed about ABC, we get $AD = CD$ and $BD = CD$, whence $\angle MAD = \angle MCD$ and $\angle DBN = \angle DCN$. The angles $\angle MAD$ and $\angle MBD$ subtend the same arc BD, They are equal, because neither A nor B can be on the arc MDN. Therefore

$$\begin{aligned} \angle MBN &= \angle MBD + \angle DBN = \angle MAD + \angle DCN \\ &= \angle MCD + \angle DCN = \angle MCN. \end{aligned}$$

14. No, it is impossible. After each step of the procedure the area and the perimeter of the polygon remain unchanged. But if a triangle and a square have equal perimeters then the area of the triangle is less than the area of the square. Indeed, it is a classical well-known result that among all triangles with perimeter P the equilateral triangle has the maximal area, which is $P^2/12\sqrt{3} < P^2/20$. The square of the same perimeter P has area equal to $P^2/16$.

15. This combinatorial problem requires a somewhat unusual induction. Let us denote the statement of the problem for an $n \times n$ table by $S(n)$. We have to prove that $S(n)$ is true for all n. What we will actually prove is that $S(n-2)$ implies $S(n)$. Together with the observation that $S(1)$ and $S(2)$ are true this will give us a proof by induction. You might like to think that we have two inductions here: one for the even numbers and another for the odd ones.

There is nothing to prove for $S(1)$ and $S(2)$, because for a 1×1 table the only cell is a diagonal one, and for 2×2 table the number of diagonal cells is exactly half of the number of all cells.

A cell in the table is characterized by a pair of integers (i, j), where i is the number of the row and j is the number of the column to which this cell belongs.

Firstly, let us note that in one case we do not need the induction. Consider all pairs of cells which are symmetric about the main diagonal; they have coordinates (i, j) and (j, i) for some $i \neq j$. If in each such pair at least one of the two cells is covered, then the minors jointly cover at least $n + (n^2 - n)/2 = (n^2 + n)/2 > n^2/2$ cells and $S(n)$ is true.

Let us assume now that $S(n-2)$ is true, and consider an $n \times n$ table in an attempt to prove $S(n)$. We only have to consider the case in which for some i, j neither cell (i, j) nor cell (j, i) is covered by the minors. Delete the ith and jth rows and the ith and jth columns and reduce the minors correspondingly if they contained some of these rows or columns. We obtain an $(n-2) \times (n-2)$ table and several reduced minors which again jointly cover the main diagonal. No reduced minor could contain both the ith row and the jth column simultaneously (otherwise this minor would cover the cell (i, j)). Similarly no reduced minor could contain simultaneously both the jth row and the ith column. Therefore the semiperimeter of a reduced minor is not less than $n-2$. Now we can apply the induction hypothesis to show that the reduced minors cover at least half of all cells of the reduced table.

To complete the induction, it is sufficient to prove that among the deleted $4n-4$ cells at least $2n-2$ were covered by the minors. Consider the minor that covers the cell (i, i). Since its semiperimeter is at least n, it covers at least $n-1$ of the deleted cells in the ith row and ith column. The minor that covers (j, j) (It may be the same minor!) also covers $n-1$ deleted cells in jth row and in jth column. But since (i, j) and (j, i) are not covered, the two minors jointly cover at least $2(n-1)$ cells. This completes the proof.

16. Let us consider the polynomial

$$P(x) = (a_1 + x)(a_2 + x) \dots (a_{100} + x) - 1.$$

Since the product of all numbers in each column is 1, the numbers $b_1, b_2, \dots, b_{100}$ are the roots of $P(x)$. Hence

$$P(x) = (x - b_1)(x - b_2) \dots (x - b_{100}),$$

and the identity

$$(a_1 + x)(a_2 + x) \dots (a_{100} + x) - 1 = (x - b_1)(x - b_2) \dots (x - b_{100})$$

holds. If we substitute $x = -a_i$ into this identity we get the equation

$$-1 = (-1)^{100}(a_i + b_1)(a_i + b_2)\dots(a_i + b_{100}),$$

which shows that the product of all numbers in the ith row is equal to -1, as was to be proved.

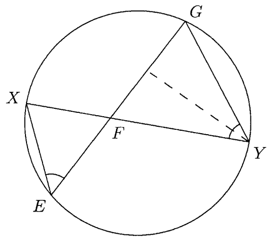
17. Answer: no, it cannot be chosen.

If the last digit α of the number a_{n-1} does not exceed 5, then $10a_n = a_{n-1} - \alpha \leq a_{n-1}$. Otherwise, if $\alpha > 5$, then the last digit of the number $a_n = 9a_{n-1} = 10a_{n-1} - a_{n-1}$ is $10 - \alpha$, which does not exceed 5. Therefore $10a_{n+1} \leq a_n = 9a_{n-1}$. In this case $a_{n+1} < a_{n-1}$. Thus in the first case we have $a_n < a_{n-1}$ and in the second we have $a_{n+1} < a_{n-1}$. If the sequence were infinite we could find an infinite strictly decreasing subsequence in it, which is impossible. Alternatively, we could assume that a_n is the smallest number in the sequence and get a contradiction.

18. Answer: $\alpha + \beta = 2$.

The left-hand sides of the given equations are the values of the polynomial $f(x) = x^3 - 3x^2 + 5x = (x-1)^3 + 2(x-1) + 3$ evaluated at the points $x = \alpha$ and $x = \beta$. Consider the function $g(y) = y^3 + 2y$. It is strictly increasing since $g'(y) = 3y^2 + 2 > 0$. Therefore the numbers α and β are uniquely determined by the equations $g(\alpha - 1) = f(\alpha) - 3 = -2$ and $g(\beta - 1) = f(\beta) - 3 = 2$. Since the function $g(x)$ is odd, $\alpha - 1 = -(\beta - 1)$, whence $\alpha + \beta = 2$.

19. Let G be the point of intersection of the ray EF with the sphere. Let XY be either AB or CD, with $X = A$ or $X = C$ (see the diagram).

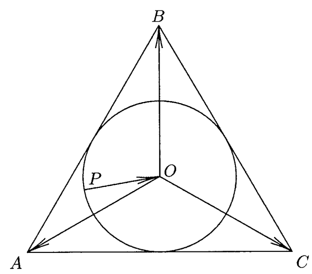


We shall prove that Y lies in the plane which is perpendicular to FG and passes through the midpoint of FG. Indeed, the points X, E, Y, G are concyclic, and the triangles XFE and YFG are similar, since $\angle FXE = \angle FGY$ and $\angle XEF = \angle GYF$ (the angles in each pair subtends the same arc). Therefore, $XE = EF$ implies $FY = YG$ and the triangle FYG is isosceles. Thus the foot of the altitude drawn from the vertex Y to the side FG coincides with the midpoint of FG.

Since this is true for $Y = B$ and for $Y = D$ we conclude that BD is perpendicular to GF and hence to EF.

20. Answer: (a) yes; (b) no.

(a) Let ABC be an equilateral triangle and let P be any point on the inscribed circle of ABC (see the diagram).



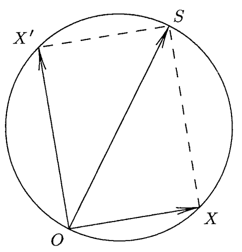
We shall prove that the 4 vectors $\overrightarrow{PO}$, $\overrightarrow{PA}$, $\overrightarrow{PB}$, $\overrightarrow{PC}$ satisfy the required equations. Let us denote: $\vec{r} = \overrightarrow{PO}$, $\vec{a} = \overrightarrow{OA}$, $\vec{b} = \overrightarrow{OB}$, $\vec{c} = \overrightarrow{OC}$. Then $|\vec{a}| = |\vec{b}| = |\vec{c}| = 2|\vec{r}|$. Computing the scalar product of $\overrightarrow{PO} + \overrightarrow{PA}$ and $\overrightarrow{PB} + \overrightarrow{PC}$ we get

$$\begin{aligned}(\overrightarrow{PO} + \overrightarrow{PA}, \overrightarrow{PB} + \overrightarrow{PC}) &= (2\vec{r} + \vec{a}, 2\vec{r} + \vec{b} + \vec{c})\\ &= (2\vec{r} + \vec{a}, 2\vec{r} - \vec{a}) = 4(\vec{r}, \vec{r}) - (\vec{a}, \vec{a}) = 4|\vec{r}|^2 - |\vec{a}|^2 = 0,\end{aligned}$$

hence these two vectors are orthogonal. The orthogonality of the other pairs of vectors can be established similarly.

(b) Suppose that such a collection of vectors exists. Let $\overrightarrow{OS}$ be the sum of all 91 vectors of this collection. Consider the sum $\overrightarrow{OX}$

of some 19 vectors from the collection. If $\overrightarrow{OX'}$ is the sum of the other 72 vectors, then $\overrightarrow{OX}+\overrightarrow{OX'}=\overrightarrow{OS}$ and $\overrightarrow{OX}\perp\overrightarrow{OX'}$. This means that the point X lies on the circle δ with diameter OS.



In principle, our collection may contain several equal vectors and we will soon see that this will be the main case for consideration. Let us first prove that it is impossible to choose 5 vectors $\vec{a}_1,\vec{a}_2,\vec{a}_3,\vec{b}_1,\vec{b}_2$ from the collection such that

$$\vec{a}_1\neq\vec{a}_2,\quad \vec{a}_1\neq\vec{a}_3,\quad \vec{a}_2\neq\vec{a}_3,\quad \vec{b}_1\neq\vec{b}_2.$$

(But $\vec{a}_1=\vec{b}_1$, for example is not prohibited.)

If it were possible, take any 17 additional vectors with sum denoted by $\vec{q}$. Let us consider the 6 vectors $\overrightarrow{OX_{ij}}=\vec{q}+\vec{a}_i+\vec{b}_j$, $i=1,2,3$; $j=1,2$. As we know, the points X_{ij} must lie on the circle δ, so we obtain 3 distinct chords $X_{i1}X_{i2}$, $i=1,2,3$. This contradicts the fact that the 3 vectors $\overrightarrow{X_{i1}X_{i2}}$ are equal as $\overrightarrow{X_{i1}X_{i2}}=\overrightarrow{OX_{i2}}-\overrightarrow{OX_{i1}}=\vec{b}_2-\vec{b}_1$.

It is clear then that there cannot be 5 distinct vectors in this collection. It even follows that there cannot be 4 distinct vectors $\vec{u}_1,\vec{u}_2,\vec{u}_3,\vec{u}_4$ in the collection. As there cannot be 5 distinct vectors at least one vector, say $\vec{u}_1$, must be repeated. But then the 5 vectors $\vec{u}_1,\vec{u}_2,\vec{u}_3,\vec{u}_4,\vec{u}_1$ satisfy the condition above, which is impossible.

Thereby the vectors of the collection can be divided into 3 classes C_1,C_2,C_3 with equal vectors forming each class. Suppose that the vectors in C_1,C_2,C_3 are equal to $\vec{x},\vec{y},\vec{z}$, respectively. Suppose that C_1 is the largest class of the three. Then C_1 contains at least 31 vectors. If either C_2 or C_3, say C_2, contains more than one vector, then consider the 3 vectors:

$$\overrightarrow{OW_1}=17\vec{x}+2\vec{y},\qquad \overrightarrow{OW_2}=18\vec{x}+\vec{y},\qquad \overrightarrow{OW_3}=19\vec{x}.$$

The points O,W_1,W_2,W_3 must be concyclic, but this contradicts the relation $\overrightarrow{OW_2}=\frac{1}{2}(\overrightarrow{OW_1}+\overrightarrow{OW_3})$.

Now only the two following possibilities have to be considered:

i. $C_1=\{\vec{x},\vec{x},\ldots,\vec{x}\}$ (89 times) , $C_2=\{\vec{y}\}, C_3=\{\vec{z}\}$;

ii. $C_1=\{\vec{x},\vec{x},\ldots,\vec{x}\}$ (90 times) , $C_2=\{\vec{y}\}, C_3=\emptyset$.

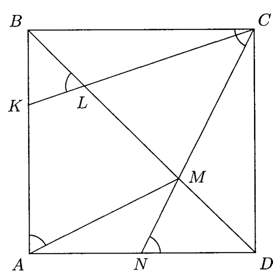
In the first case consider the vectors:

$$\overrightarrow{OY_1}=18\vec{x}+\vec{y},\ \overrightarrow{OY_2}=18\vec{x}+\vec{z},\ \overrightarrow{OY_3}=17\vec{x}+\vec{y}+\vec{z},\ \overrightarrow{OY_4}=19\vec{x}.$$

The points Y_1,Y_2,Y_3,Y_4 are again on the circle δ. The chords Y_1Y_2 and Y_3Y_4 are distinct, because $Y_3\neq Y_1$, $Y_3\neq Y_2$, but their midpoints coincide. It can be true only when Y_1Y_2 and Y_3Y_4 are both diameters of δ. But if Y_1Y_2 is a diameter, then $\overrightarrow{OY_1}+\overrightarrow{OY_2}=\overrightarrow{OS}$ so $36\vec{x}+\vec{y}+\vec{z}=89\vec{x}+\vec{y}+\vec{z}$, which is impossible.

In the other case the orthogonality conditions would be $(18\vec{x}+\vec{y},72\vec{x})=0$ and $(19\vec{x},71\vec{x}+\vec{y})=0$. These imply $18(\vec{x},\vec{x})+(\vec{x},\vec{y})=0$ and $71(\vec{x},\vec{x})+(\vec{x},\vec{y})=0$, respectively. We get $(\vec{x},\vec{x})=0$, and therefore $\vec{x}=0$, which is a contradiction.

21. Let us prove, first, that $\angle BKC+\angle DNC=\frac{3}{4}\pi$.



Suppose the sidelength of the square is equal to 1 and set

$$\alpha=BK=\cot\angle BKC,\qquad \beta=DN=\cot\angle DNC.$$

According to the conditions of the problem, we get $(1-\alpha)(1-\beta)=2\alpha\beta$ so $-(\alpha+\beta)=-1+\alpha\beta$. Since $0<\alpha<1$ and $0<\beta<1$ we can be sure that $\alpha\beta\neq1$ and therefore

$$-1=\frac{\alpha+\beta}{-1+\alpha\beta}=\tan(\angle BKC+\angle DNC),$$

which implies $\angle BKC + \angle DNC = \frac{3}{4}\pi$. Therefore

$$\angle BLK = \pi - \frac{\pi}{4} - \angle BKC = \angle DNC = \angle BCM.$$

We also note that $\angle BCM = \angle BAM$ since these angles are symmetric with respect to the diagonal BD. It follows now that $\angle KLM + \angle KAM = \pi$, and the points A, K, L, M are concyclic. It can be proved in the same way that the points A, N, M, L are also concyclic. These circles coincide as they have 3 common points.

22. Answer: 6 in both cases.

(a) Each country must belong to at least 3 alliances, otherwise in all it can have no more than $49 + 49 = 98 < 99$ allies. Hence there exist at least $\frac{3\times 100}{50} = 6$ alliances. To show that 6 alliances are indeed possible we will give an example which also satisfies the additional requirement that no pair of alliances jointly contains more than 80 countries and will therefore also answer part (b).

(b) To construct the example, let us divide all countries into 10 blocks

$$A_1, A_2, \ldots, A_5, B_1, B_2, \ldots, B_5,$$

with 10 countries in each, and form the alliances as follows:

$$\{A_1 \cup B_1 \cup B_2 \cup B_3 \cup A_3\}, \qquad \{A_2 \cup B_2 \cup B_3 \cup B_4 \cup A_4\},$$
$$\{A_3 \cup B_3 \cup B_4 \cup B_5 \cup A_5\}, \qquad \{A_4 \cup B_4 \cup B_5 \cup B_1 \cup A_1\},$$
$$\{A_5 \cup B_5 \cup B_1 \cup B_2 \cup A_2\}, \qquad \{A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5\}.$$

23. Consider the polynomial

$$f(x) = \prod_{i=1}^{n}(x + a_i) - \prod_{j=1}^{n}(x - b_j),$$

which is of degree less than n. We get

$$f(b_j) = \prod_{i=1}^{n}(a_i + b_j) = c$$

for each $j = 1, 2, \ldots, n$ and some c, which does not depend on j. The polynomial $f(x) - c$ has degree less than n, and at least n distinct roots and hence $f(x) - c \equiv 0$ identically. But then

$$c = f(-a_i) = -\prod_{j=1}^{n}(-a_i - b_j) = (-1)^{n+1}\prod_{j=1}^{n}(a_i + b_j).$$

This means that the products of the numbers in each row are all equal to $(-1)^{n+1}c$ and therefore equal to each other.

24. For $k = 1, 2, \ldots, 1990$ let us estimate $|y_k - y_{k+1}|$ as follows:

$$\begin{aligned}|y_k - y_{k+1}| &= \left|\frac{x_1 + \ldots + x_k}{k} - \frac{x_1 + \ldots + x_{k+1}}{k+1}\right| \\ &= \left|\frac{x_1 + \ldots + x_k - kx_{k+1}}{k(k+1)}\right| \\ &\le \frac{|x_1 - x_2| + 2|x_2 - x_3| + \ldots + k|x_k - x_{k+1}|}{k(k+1)}.\end{aligned}$$

Let us recall the following well-known identity:

$$\left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{(n-1)\cdot n}\right)$$
$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right) = 1 - \frac{1}{n}$$

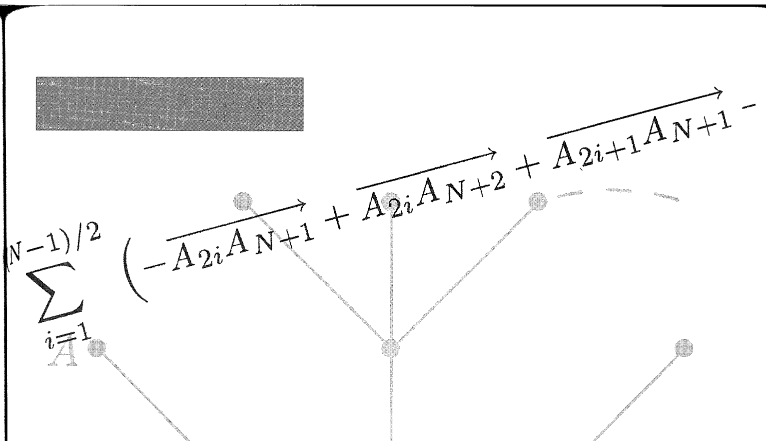
and a consequence of it

$$\left(\frac{1}{k\cdot(k+1)} + \ldots + \frac{1}{(n-1)\cdot n}\right) = \frac{1}{k}\left(1 - \frac{k}{n}\right).$$

They will lead us through the following estimate

$$\begin{aligned}&|y_1 - y_2| + \ldots + |y_{1990} - y_{1991}| \\ &\le |x_1 - x_2|\cdot\left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{1990\cdot 1991}\right) + \\ &\quad 2|x_2 - x_3|\cdot\left(\frac{1}{2\cdot 3} + \ldots + \frac{1}{1990\cdot 1991}\right) + \ldots \\ &\quad \ldots + 1990|x_{1990} - x_{1991}|\cdot\frac{1}{1990\cdot 1991} = \\ &= |x_1 - x_2|\left(1 - \frac{1}{1991}\right) + |x_2 - x_3|\left(1 - \frac{2}{1991}\right) + \ldots \\ &\quad + |x_{1990} - x_{1991}|\left(1 - \frac{1990}{1991}\right) \le 1991\cdot\left(1 - \frac{1}{1991}\right) = 1990.\end{aligned}$$

This estimate is exact for $x_1 = 1991$, $x_2 = \ldots = x_{1991} = 0$.



$\sum_{i=1}^{(N-1)/2} \left(-\overrightarrow{A_{2i}A_{N+1}} + \overrightarrow{A_{2i}A_{N+2}} + \overrightarrow{A_{2i+1}A_{N+1}} -\right.$
A
B
X

CIS OLYMPIAD 1992

9 FORM

First day

1. Prove that for all positive numbers a, b, c the following inequality holds
$$a^4 + b^4 + c^2 \geq 2\sqrt{2}abc.$$
(V Radchenko, Kiev)

2. Let E be a point on the diagonal BD of a square $ABCD$ and let O_1 and O_2 be the circumcentres of the triangles ABE and ADE respectively. Prove that the quadrilateral AO_1EO_2 is a square.
(S Anisov, Moscow; S Rukshin, St. Petersburg)

3. Towns of an empire with k large cities are connected by roads so that each town is accessible from any other. Prove that this empire can be divided into k republics in such a way that each republic has a large city as its capital and so that each republic contains at least one shortest path from each of its towns to its capital. (A shortest path between two towns is a path passing through as few towns as possible.)
(A Perlin, Moscow)

4. An infinite board is divided into cells by horizontal and vertical lines. Initially some cells are occupied by pieces. We say that two pieces are neighbours if they occupy two cells with a side in common. A move consists of jumping a piece over one of its neighbours into the cell beyond, if this cell is empty, and removing the piece which was jumped over. Let $m > 1$ and $n > 1$ be integers. What is the smallest number of pieces that can be left on the board, if the pieces initially occupied all of the cells of an $m \times n$ rectangle and the other cells are empty?
(I Solov'ev, Moscow)

9 FORM

Second day

5. Does there exist a 4-digit integer (in decimal form) such that no replacement of three its digits by another three gives a multiple of 1992?

(I Selishev, S Konyagin, Moscow)

6. Points A and B are situated on a circle with a centre O. A point P is chosen on the smaller arc AB, the points Q and R are the symmetric points to P relative to OA and OB, respectively. Prove that the point P' of intersection of the segments AR and BQ is symmetric to P with respect to AB.

(V Proizvolov, B Kukushkin, Moscow)

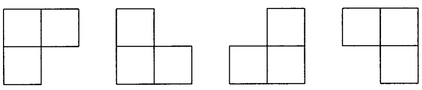
7. Solve the system of equations

$$\begin{cases} (1+x)(1+x^2)(1+x^4) = 1+y^7, \\ (1+y)(1+y^2)(1+y^4) = 1+x^7 \end{cases}$$

for real x and y.

(D Mit'kin, Moscow)

8. An $m \times n$ rectangle is divided into mn unit squares by horizontal and vertical lines. Do there exist natural numbers m and n such that the rectangle can be covered without overlapping by "gnomons" as



shown in the diagram in a such a way that no two "gnomons" form a 2×3 or 3×2 rectangle and no point is a vertex of more than three "gnomons."

(B Kukushkin, Moscow)

10 FORM

First day

9. Prove that for any $a > 1, b > 1$ the following inequality holds

$$\frac{a^2}{b-1} + \frac{b^2}{a-1} \geq 8.$$

(V Radchenko, Kiev)

10. Prove that among any 15 pairwise coprime natural numbers, all of which are greater than 1 but not greater than 1992, there exists at least one prime number.

(D Tereshin, Moscow)

11. In a movie-theatre there are m columns and n rows of seats. An absent-minded ticket seller sold $m \times n$ tickets so that some of the same seats were sold several times but some other seats were left unsold. An usher managed to allocate the seats in such a way that each ticket-holder got a seat in either the same row or the same column as was shown on the ticket. Let us call such an allocation of seats clever.

(a) Prove that there exists a clever allocation of seats such that at least one ticket-holder occupies the correct seat;

(b) What is the maxumum number of ticket-holders that can be always offered their own seats under a clever allocation of seats?

(E Malinnikova, St. Petersburg)

12. Three circles S_1, S_2, S are given in the plane. The circles S_1 and S_2 both intersect the circle S and pass through the centre O of S. The second point of intersection of S_1 and S_2 is denoted by M. Let C, D and A, E be the points of intersection of S with S_1 and S_2, respectively. Suppose that the lines AD and CE intersect at a point B, different from M. Prove that $\angle BMO = 90°$.

(L Kuptzov, Moscow)

10 FORM

Second day

13. The sequence of positive integers (x_n) is defined as follows:

$$x_1 = 1, \qquad x_{n+1} = n + x_1^2 + \ldots + x_n^2.$$

Prove that there are no squares of natural numbers in this sequence except x_1.

(A Perlin, St. Petersburg)

14. A parallelogram $ABCD$ is given in the plane. A circle with centre P touches side BC and the continuations of the side AB and the diagonal AC. A circle with a centre Q touches the side CD and the continuations of the side AD and the diagonal AC. Let K be the point of tangency of the first circle with the line AB, L be the point of tangency of the second circle with the line AD, M be the point of intersection of the lines AB and QC, and N be the point of intersection of the lines AD and PC. Prove that $KM = NL$.

(D Tereshin, Moscow)

15. Each cell of a rectangular $2m \times n$ grid is coloured with one of the two colours, blue or green, so that half of the cells are coloured blue and half of the cells are coloured green. Suppose further that the left lower cell is blue and the upper right cell is green. The centres of the blue cells are connected by straight line segments, as are the centres of the green cells. Prove that it is possible to turn these segments into vectors by putting arrow-heads on them in such a way that the sum of the vectors is 0.

(N Agahanov, Moscow)

16. Prove that in a company of 17 persons, in which each person is acquainted with exactly 4 other persons, there exist two, who are not acquainted and have no mutual acquaintance.

(S Duzhin)

11 FORM

First day

17. Find all zeros of the function

$$f(x) = a\cos(x+1) + b\cos(x+2) + c\cos(x+3),$$

given that the coefficients a, b, c are chosen so that there are at least 2 zeros of this function in the interval $(0, \pi)$.

(I Voronovich, Minsk)

18. A circle ρ is the intersection of a plane and a sphere with centre O. Two points Let A and B be points on the sphere on either side of the plane. The radius OA is perpendicular to the plane. Another plane π is drawn through the line AB, and it intersects the circle ρ at points X and Y. Prove that the product $BX \cdot BY$ does not depend on the choice of π.

(B Chinik, Kishinev)

19. Imagine that you possess a procedure for determining all real zeros of any cubic polynomial $P(x) = ax^3 + bx^2 + cx + d$. Find an algorithm for solving the system

$$\begin{cases} x = P(y), \\ y = P(x) \end{cases}$$

by using this procedure.

(D Tulyakov, Moscow)

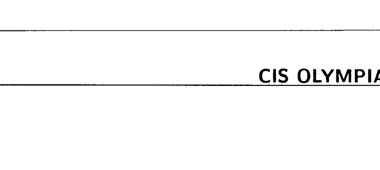
20. Determine all natural numbers $k > 1$ such that, for some distinct natural numbers m and n, the numbers $k^m + 1$ and $k^n + 1$ can be obtained from each other by reversing the order of the digits in their decimal representations.

(A Skopenkov, Moscow)

11 FORM

Second day

21. An equilateral triangle, with the length of its sides equal to 10, is divided by straight lines parallel to its sides into 100 equal equilateral triangles with the length of their sides equal to 1. There are m tiles of the first kind shown on the left diagram and $25 - m$ tiles of the second kind shown on the right diagram:



(a) Is it possible to cover the original triangle with the given tiles if $m = 10$?

(b) Find all values of m for which it is possible to cover the original triangle with the given tiles.

(E Malinnikova, St. Petersburg)

22. 1992 vectors are given in the plane. Two players are playing a game. The first player makes the first move and then they take turns to make moves. In one move the first player can paint red any one of those vectors which have not yet been painted. The second player makes similar moves but paints vectors green. At the end of the game, when all vectors are painted, the red vector, which is the sum of all painted red vectors, and the green vector, which is the sum of all painted green vectors, are computed. If the red vector is longer than the green one, the first player wins. Is it true that the first player can play so as not to lose?

(I Voronovich, Minsk)

23. Prove that for natural numbers $k < l < m < n$, satisfying $kn = lm$, the following inequality holds

$$\left(\frac{n-k}{2}\right)^2 \geq k+2.$$

(E Malinnikova, St. Petersburg)

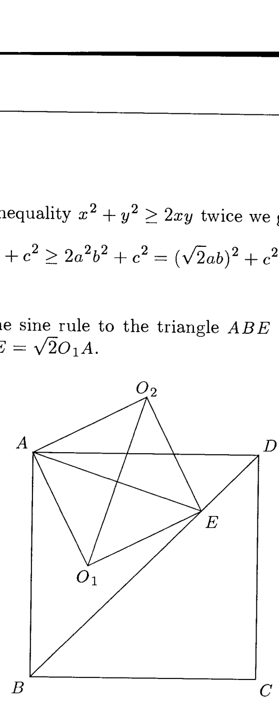
24. See Question 16.

SOLUTIONS

1. Applying the inequality $x^2 + y^2 \geq 2xy$ twice we get

$$(a^4 + b^4) + c^2 \geq 2a^2b^2 + c^2 = (\sqrt{2}ab)^2 + c^2 \geq 2\sqrt{2}abc.$$

2. By applying the sine rule to the triangle ABE we obtain $AE = 2O_1A \sin \angle ABE = \sqrt{2}O_1A$.



Similarly, $AE = \sqrt{2}O_2A$. Hence $O_1E = O_1A = O_2A = O_2E$, and the quadrilateral AO_1EO_2 is a rhombus. It remains only to note that $\angle AO_1E = 2\angle ABE = 90°$.

3. We shall say that the distance between towns A and B is equal to k if it is possible find a path from A to B passing through $k - 1$ other towns but there is no such path passing through fewer towns. We write $d(A, B) = k$ to denote this.

Let us start partitioning by declaring the large cities of the empire to be the capitals of newborn republics. Then we classify all towns, which are within distance 1 from the capitals. If a town A is within distance 1 from a capital C_i, and within distance 2 or more from other capitals, then we include A in C_1's republic R_i. If the town A is within distance 1 of several capitals, we choose any one of them, and include A in the corresponding republic. At step k we classify those towns, which have not yet been classified, and which are within distance 1 of at least one classified town. If a town

A is within distance 1 of the towns $B_1, \dots, B_s$ of the republics $R_1, \dots, R_s$, then we randomly assign A to one of the republics $R_1, \dots, R_s$.

We prove by induction that if the distance from a town A to the nearest large city is equal to k, then A will be classified at step k. Indeed, there exists a path of length k from some large city L to A:

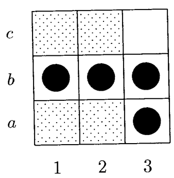
$$L = A_0, A_1, \dots, A_{k-1}, A_k = A$$

such that $d(A_{i-1}, A_i) = 1$. It is clear that $L = A_0, A_1, \dots, A_{k-1}$ is the shortest path from A_{k-1} to one of its nearest large cities, namely L. Therefore by the induction hypothesis A_{k-1} will be classified at step $k-1$, and hence A will be classified at step kth.

Suppose now that the distance from a town A to the nearest large city is equal to k and that L_i is the capital of the republic, to which A was assigned. We know that A was classified at the kth step. It follows from the way in which the partition was constructed that there exists a path of length k from L_i to A. This path, as we know, has the shortest length.

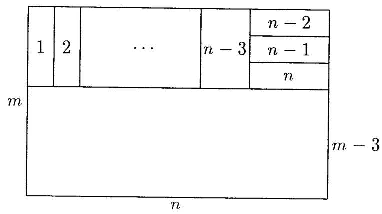
4. **Answer:** 2, if mn is divisible by 3, and 1 otherwise.

First, we will give an algorithm for reducing the number of pieces on the board to the required number. The key observation is that in the fragment shown below

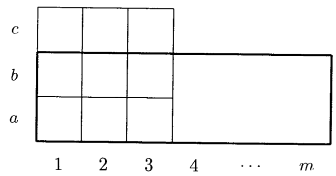


with the empty cell $c3$ (the cells $a1$, $a2$, $c1$, $c2$ can be either occupied or not), we can remove from the board the horizontal triple $b1$, $b2$, $b3$ by moves $a3 : b3$, $b1 : b2$, $c3 : b3$. We will use this observation to transform an $m \times n$ rectangle into an $(m-3) \times n$ rectangle, with $m \geq 4$ and $n \geq 2$. For $n \geq 3$ we can remove the n triples shown in

the diagram in the order specified:

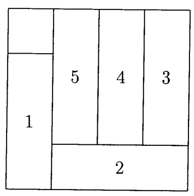


If $n = 2$ we can obtain an $(m-3) \times n$ rectangle by removing two triples $a1, a2, a3$ and $b1, b2, b3$, shown in the diagram,



by the moves $a1 : b1$, $a2 : b2$, $c1 : c2$ and further removing the triple $a3, b3, c3$ as before.

Therefore any $m \times n$ rectangle can be transformed into one of the six types of rectangles rectangles: 1×2, 2×2, 4×4, 1×3, 2×3, 3×3. (Note that we do not transform a 4×4 rectangle into a 1×4 strip!) For the first two the reduction to one piece is clear. For the last three it is easy to find an algorithm, which reduces the number of pieces to 2. For the 4×4 square we can remove 5 triples in the order shown in the following diagram:



Let us now prove that it is impossible to reduce the number of pieces any further. Obviously, it is impossible to get less than one piece. Therefore we only have to consider the case when mn is divisible by 3. Let us colour the diagonals of the board with three colours, as shown in the diagram.

	1	2	3	1	2	3	
	3	1	2	3	1	2	
	2	3	1	2	3	1	
	1	2	3	1	2	3	
	3	1	2	3	1	2	
	2	3	1	2	3	1	

Any 3×1 strip contains exactly one cell of each colour. Therefore the $m \times n$ rectangle contains an equal number $\frac{1}{3}mn$ cells of each colour. Suppose now that after several moves i, j, k are the numbers of cells of the first, second and the third colour, respectively. Then after the next move the numbers (i, j, k) will be changed into numbers $(i-1, j-1, k+1)$, $(i-1, j+1, k-1)$, or $(i+1, j-1, k-1)$. Since at the beginning the parities of all numbers were the same, these parities will be the same after every move. Hence it is impossible to get only one piece at the end. (And of course the last two pieces will be of the same colour.)

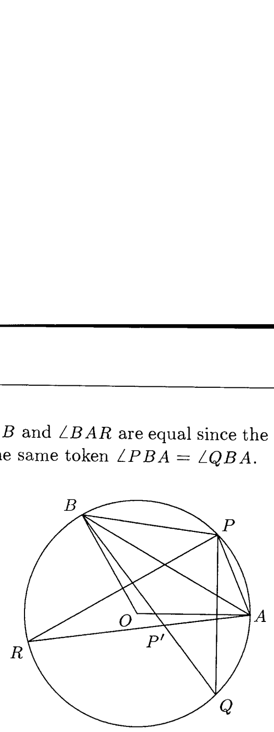
5. **Answer:** yes, such a number exists.

There are only 6 four-digit nonnegative integers, which are multiples of 1992:

$$0000, \quad 1992, \quad 3984, \quad 5976, \quad 7968, \quad 9960.$$

Choose an arbitrary number, whose ith digit is different from the ith digits of the 6 listed integers, for instance, 2111. Clearly, it will be a number of the type required.

6. The angles $\angle PAB$ and $\angle BAR$ are equal since the arcs BP and BR are equal. By the same token $\angle PBA = \angle QBA$.



Hence the triangles APB and $AP'B$ have a common side AB and two equal angles. Therefore they are congruent and the points P and P' are symmetric with respect to AB.

7. **Answer:** the system has two solutions: $(0, 0)$ and $(-1, -1)$.

Note first that if $x \neq 1$, then

$$(1+x)(1+x^2)(1+x^4) = \frac{(1-x^8)}{(1-x)}.$$

If $x = y$, then $x \neq 1$ and $(1-x^8)/(1-x) = 1 + x^7$, from which we deduce that $x^7 = x$, and hence $x = 0$ or $x = -1$. Thus we obtain the two solutions in the answer. Let us prove that there are no solutions if $x \neq y$.

If $x = 0$, then from the first equation it is clear that $y = 0$ as well. If $x > 0$ and $y < 0$, then $(1+x)(1+x^2)(1+x^4) > 1$ and $1 + y^7 < 1$, under which conditions the first equation cannot be satisfied. The case $x < 0$ and $y > 0$ is similar. Let $x > 0$ and $y > 0$. For any positive x we have the following inequality $(1+x)(1+x^2)(1+x^4) > 1 + x^7$. Therefore, if $x > y$, then $(1+x)(1+x^2)(1+x^4) > 1+x^7 > 1+y^7$, and the first equation cannot be satisfied. The same argument shows that if $y > x > 0$ the second equation cannot be satisfied.

Suppose that $x < 0$ and $y < 0$. If $x > y$, then $x^7 > y^7$, $x^6 < y^6$, $x^8 < y^8$, $xy > 0$. Multiplying the first equation by $1-x$, the second equation by $1-y$, and subtracting one from the other we get

$$y^8 - x^8 = y - x + y^7 - x^7 + xy(x^6 - y^6).$$

This equation cannot be satisfied, since the left-hand side is positive while the right-hand side is negative. The case $y > x$ is similar.

8. **Answer:** No, there do not.

Suppose, on the contrary, that there are such numbers m and n for which the $m \times n$ rectangle can be covered as required. If in this cover some "gnomon" touches the border of the rectangle along the side of only one cell, it must necessarily be supplemented by another "gnomon" so that they form a 3×2 rectangle. Thus each "gnomon," which touches the border, touches it along a side of length 2. Hence the rectangle has even sidelengths, say $2p \times 2q$.

Note that each "gnomon" has 6 vertices so the "gnomons" in the rectangle have $\frac{2p \times 2q}{3} \times 6 = 8pq$ vertices in total. Let us now compute the number of vertices in a different way. At every gridpoint, which is not on the border, according to the conditions of the problem there can be no more than two vertices. Let us now consider the intersections on the border. At the intersections on a side of the rectangle of length $2p$, different from the corners, there are no more than $2(p-1)$ vertices in total, and at each corner of the rectangle there is only one vertex of some "gnomon." In total there are no more than

$$2 \times (2p-1)(2q-1) + 2 \times 2(p-1) + 2 \times 2(q-1) + 4 = 8pq - 2$$

vertices, which is less than the $8pq$ obtained in the first count. This contradiction completes the proof.

9. Using the Arithmetic mean - Geometric mean inequality we get

$$\frac{a^2}{b-1} + \frac{b^2}{a-1} \geq 2\sqrt{\frac{a^2}{b-1} \cdot \frac{b^2}{a-1}} = 2 \cdot \frac{a}{\sqrt{a-1}} \cdot \frac{b}{\sqrt{b-1}}.$$

It remains only to notice that for $x > 1$ the inequality $x/\sqrt{x-1} \geq 2$ holds, since for $x > 1$ it is equivalent to $x^2 \geq 4(x-1)$, or $(x-2)^2 \geq 0$.

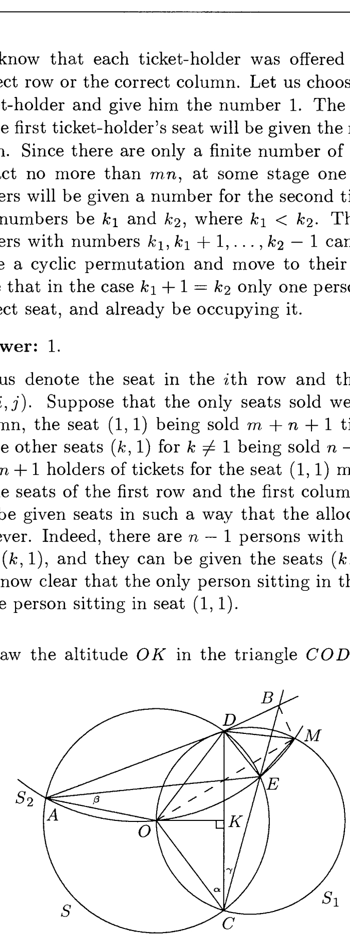
10. Let $n_1, n_2, \ldots, n_{15}$ be 15 pairwise coprime positive integers each greater than 1. Suppose that all of them are composite. Let p_k be the smallest prime divisor of n_k. Since the numbers $n_1, n_2, \ldots, n_{15}$ are coprime, the primes $p_1, p_2, \ldots, p_{15}$ are distinct, and without loss of generality we may assume that $p_1 < p_2 < \ldots < p_{15}$. Then $p_{15} \geq 47$ and $n_{15} \geq 47^2 > 1992$, which is a contradiction.

11. (a) We know that each ticket-holder was offered a seat in the correct row or the correct column. Let us choose an arbitrary ticket-holder and give him the number 1. The person sitting in the first ticket-holder's seat will be given the number 2, and so on. Since there are only a finite number of ticket-holders, in fact no more than mn, at some stage one of the ticket-holders will be given a number for the second time. Let these two numbers be k_1 and k_2, where $k_1 < k_2$. Then the ticket-holders with numbers $k_1, k_1 + 1, \ldots, k_2 - 1$ can be offered to make a cyclic permutation and move to their correct seats. Note that in the case $k_1 + 1 = k_2$ only one person will get the correct seat, and already be occupying it.

(b) **Answer:** 1.

Let us denote the seat in the ith row and the jth column by (i, j). Suppose that the only seats sold were in the first column, the seat $(1, 1)$ being sold $m + n + 1$ times and each of the other seats $(k, 1)$ for $k \neq 1$ being sold $n - 1$ times. The $m + n + 1$ holders of tickets for the seat $(1, 1)$ must occupy all of the seats of the first row and the first column. The others can be given seats in such a way that the allocation of seats is clever. Indeed, there are $n - 1$ persons with tickets for the seat $(k, 1)$, and they can be given the seats $(k, 2), \ldots, (k, n)$. It is now clear that the only person sitting in the correct seat is the person sitting in seat $(1, 1)$.

12. Let us draw the altitude OK in the triangle COD (see the diagram).



Since $\angle BMO = \angle BMD + \angle DMO$, the statement will be proved if we establish that

(a) $\angle DMO = \angle KCO$,

(b) $\angle BED = \angle COK$,

(c) $\angle BMD = \angle BED$.

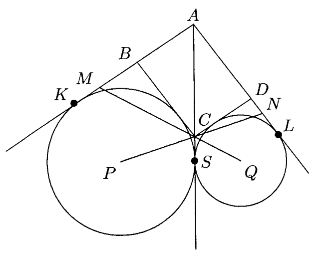
Equation (a) follows from the fact that $\angle DMO$ and $\angle KCO = \angle DCO$ are inscribed angles both subtending the arc DO of the circle S_1. Since the quadrilateral $DECA$ is inscribed in S, we get $\angle BED = \pi - \angle DEC = \angle DAC = \frac{1}{2}\angle DOC = \angle COK$ and equation (b) is proved.

Let us prove (c). For this it is sufficient to prove, that the points B, M, E, D, are concyclic or, equivalently that $\angle DME = \angle DBE$. We shall show that both these angles are equal to $\alpha + \beta$, where $\alpha = \angle DCO$, $\beta = \angle OAE$. Due to the property of inscribed angles, $\angle DME = \angle DMO + \angle OME = \angle DCO + \angle OAE = \alpha + \beta$.

The angle $\angle ADC$ is an exterior angle of the triangle CDB, thus $\angle ADC = \angle DBC + \angle BCD = \angle DBE + \gamma$, where $\gamma = \angle BCD = \angle ECD = \angle EAD$. On the other hand $\angle ADC = \angle ADO + \angle ODC = \angle OAD + \angle DCO = \angle OAE + \angle DAE + \angle DCO = \beta + \gamma + \alpha$. Thus $\angle DBE + \gamma = \angle ADC = \beta + \gamma + \alpha$, and so $\angle DBE = \alpha + \beta$. It follows that $\angle DBE = \alpha + \beta = \angle DME$ and equation (c) holds.

13. It follows that $x_2 = 2$ and $x_{n+1} = x_n^2 + x_n + 1$ for $n \geq 2$. Thus for $n \geq 1$ the inequality $x_n^2 < x_{n+1} < (x_n + 1)^2$ holds, showing that for $n \geq 1$ the number x_{n+1} is squeezed between two consecutive squares, and therefore is not a square of itself.

14. We start with a proof that both circles touch the line AC at the same point S.



Suppose that the circle with centre P touches AC at a point S_1. Then, since two tangents to a circle drawn from the same point are equal in length, we get $BC = KB + CS_1$. Therefore

$$AK + AS_1 = KB + AB + AC + CS_1 = AB + BC + AC$$

and since $AK = AS_1$ we get $AS_1 = \frac{1}{2}(AB + BC + AC)$. Similarly the circle with the centre Q touches AC at a point S_2 with $AS_2 = \frac{1}{2}(AD + CD + AC)$. Since $AD = BC$ and $AB = CD$ we obtain that $S_1 = S_2$, and we denote this point by S.

Since the bisectors of the adjacent angles SCD and ACD are perpendicular to each other and the bisectors of the angles ACD and BAC are parallel, we conclude that CQ is perpendicular to AP. Hence the triangle AMC is isosceles with $AM = AC$. By the same token, $AN = AC$ and thus $AM = AN$. The following tangents are equal: $AK = AS = AL$. Hence $KM = AK - AM = AL - AN = NL$.

15. Let $N = mn$ and consider separately two possibilities: (a) N is odd and (b) N is even.

(a) We shall call a segment blue if it connects the centres of two blue cells and green if it connects the centres of two green cells. We are going to prove that it is possible to put arrow-heads on the blue segments in such a way that the sum of all blue vectors so obtained is equal to zero. It will be clear that the same can be done with the green segments.

Let us denote the centres of the blue cells by $A_1, \ldots, A_N$, and let us put arrow-heads on them according to the following rule: the segment A_nA_{n+k} becomes the vector $\overrightarrow{A_nA_{n+k}}$, if k is odd, and $\overrightarrow{A_{n+k}A_n} = -\overrightarrow{A_nA_{n+k}}$, if k is even. We prove by induction that the sum of all such vectors is zero. This is true for $N = 3$, which is the base of the induction. Since we prove the statement for all odd numbers N, the induction step must prove that if the statement is true for some N, it is also true for $N + 2$. Therefore to prove the induction step we have to add two cells A_{N+1} and A_{N+2}. This adds the following sum of vectors:

$$\left(\overrightarrow{A_1A_{N+1}} - \overrightarrow{A_1A_{N+2}} + \overrightarrow{A_{N+1}A_{N+2}}\right) +$$

$$\sum_{i=1}^{(N-1)/2} \left(-\overrightarrow{A_{2i}A_{N+1}} + \overrightarrow{A_{2i}A_{N+2}} + \overrightarrow{A_{2i+1}A_{N+1}} - \overrightarrow{A_{2i+1}A_{N+2}}\right),$$

which is zero.

(b) Let us denote by A and B the centres of the lower left and upper right cells, which are respectively blue and green. Let

$$A_1, \ldots, A_{N-1}$$

and

$$B_1, \ldots, B_{N-1}$$

be the centres of the other blue and green cells, respectively. Let us put arrow-heads on the blue segments connecting

$$A_1, \ldots, A_{N-1}$$

as in (a) and similarly for the green segments connecting

$$B_1, \ldots, B_{N-1}.$$

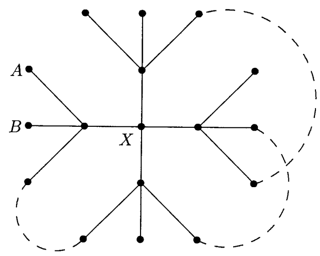
In either group the sum of vectors is zero.

Let C and D be the centres of two arbitrary cells, which are symmetric with respect to the centre of the rectangle. If they are different colours, say C is blue and D is green, then $\overrightarrow{AC} + \overrightarrow{BD} = \vec{0}$, and we can put arrow-heads on the segments AC and BD so that the sum of the two vectors obtained is zero. If they are the same colour, say blue, we put arrow-heads on the segments AC and AD so as to obtain vectors $\overrightarrow{AC}$ and $\overrightarrow{AD}$, whose sum is $\overrightarrow{AC} + \overrightarrow{AD} = \overrightarrow{AB}$. If $C'$and D' are the centres of a pair of symmetric green cells, we put arrow-heads on the segments BC' and BD' to obtain vectors $\overrightarrow{BC'}$ and $\overrightarrow{BD'}$ with sum $\overrightarrow{BA} = -\overrightarrow{AB}$. Since the number of blue symmetric pairs is equal to the number of green symmetric pairs, the total sum will be zero.

16. Let us draw a graph, whose vertices correspond to the 17 persons and whose edges correspond to their acquaintances. We say that the distance between two vertices A and B of the graph is k if A and B can be connected by a path containing k edges but there is no path connecting A to B which contains fewer than k edges.

We have to disprove that any two vertices of the graph are within distance 2 of each other. Suppose on the contrary that every vertex X can be connected to any of the other 16 vertices, either by a single edge or by two edges via some third vertex. The vertex X is connected by a single edge to exactly 4 other vertices, and each of them, in its turn, is connected to X and 3 others. Let us consider

all of these vertices. There can be no more than 17 such vertices in total (see the diagram).



By doing this, we get all vertices of the graph which are within distance ≤ 2 of the vertex X. Therefore the 17 vertices in the diagram are in fact all of the vertices of the graph and all 17 are distinct. Since X was arbitrary we have also proved that no two neighbours of any vertex are connected by an edge. For example the points A and B in the diagram are not connected by an edge. The other edges of the graph, and there are $\frac{17\times 4}{2} - 16 = 18$ of them, must connect the 12 vertices, at distance 2 from X subject to this restriction. (Some of these hypothetical edges are shown in the diagram by dotted lines.) Each of these 18 edges completes a cycle of 5 edges passing through X. Since X is arbitrary, we may conclude that through each of the other 16 vertices there also passes exactly 18 such cycles. Each cycle passes through 5 vertices, thus the total number of cycles must be $\frac{18\times 17}{5}$. But this is not an integer.

17. Using the formula

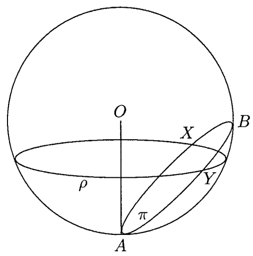
$$\cos(x+d) = \cos d \cos x - \sin d \sin x$$

to express $\cos(x+d)$ for $d = 1, 2, 3$, we obtain that

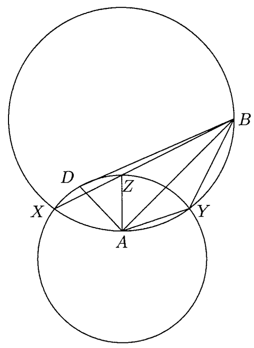
$$f(x) = A\cos x + B\sin x = C\cos(x+\phi)$$

for some constants A, B, C, ϕ. Since for $C \neq 0$ the distance between two consecutive zeros of the function $C\cos(x+\phi)$ is equal to π, the condition of the problem can only be satisfied if $C = 0$. In this case every real number is a zero of the function.

18. The intersection of the sphere and the plane π is a circle that contains the points A, B, X, Y. The values $AB = a$, $AX = AY = b$ do not depend on π and, moreover, $AX = AY$ implies $\angle ABX = \angle ABY$.



In the plane π let us draw the circle with centre A and radius b.



It intersects the larger of the segments between BX and BY, say BX, at a point Z. Since $\angle ABX = \angle ABY$ the points Y and Z are equidistant from B, that is $BZ = BY$. (If $BX = BY$, we consider $Z = X$.) Let D be the point of tangency of a tangent to this circle drawn from B. Then

$$BX \cdot BY = BX \cdot BZ = BD^2 = AB^2 - AD^2 = a^2 - b^2,$$

which is a constant.

19. The sum and the difference of the two original equations form the system

$$\begin{cases} a(x^3 + y^3) + b(x^2 + y^2) + (c - 1)(x + y) + 2d = 0, \\ (x - y)\left[a(x^2 + xy + y^2) + b(x + y) + (c + 1)\right] = 0. \end{cases}$$

Solutions satisfying $x = y$ are to be found by solving the cubic equation $P(x) - x = 0$. For the other solutions, let $u = x + y$ and $v = xy$. Then $x^2 + y^2 = u^2 - 2v$, $x^3 + y^3 = u^3 - 3uv$, and we obtain the system

$$\begin{cases} a(u^3 - 3uv) + b(u^2 - 2v) + (c - 1)u + 2d = 0, \\ a(u^2 - v) + bu + (c + 1) = 0. \end{cases}$$

If we find v in terms of u from the second equation and substitute it into the first equation, we obtain a cubic equation in u. Solving this we find all pairs (u, v) and then all pairs (x, y).

20. Answer: $k = 3$.

The solution makes use of the following well-known problem. We formulate it as a lemma and leave it to the reader.

Lemma. Suppose that two numbers M and N can be obtained from each other by reversing the order of the digits in their decimal representations. Then their difference $M - N$ is a multiple of 9.

Suppose that for some $m < n$ the numbers $k^m + 1$ and $k^n + 1$ can be obtained from each other by reversing the order of the digits in their decimal representations. Then they have the same number of digits and therefore $10(k^m + 1) > k^n + 1$.

For $k > 10$ this inequality does not hold, since in this case

$$\begin{aligned} k^n + 1 \;&\geq\; k \cdot k^m + 1 = k(k^m + 1) - k + 1 \\ &\geq\; 10(k^m + 1) + (k^m + 1) - k + 1 > 10(k^m + 1) \end{aligned}$$

as $k^m \geq k$. It is also clear that $k \neq 10$.

The case $2m < n$ is impossible, because

$$k^n + 1 \geq k^m \cdot k^{m+1} > k^m(k^m + 1),$$

and $k^m < 10$. In this case k^m is a 1-digit number. Since reversing the order of the digits does not change 1-digit numbers and $k^m + 1 \neq 10$, this implies $k^m + 1 = k^n + 1$ which contradicts $m < n$.

Hence $2m \geq n$. Then $m \geq n - m$ and it is easy to see that

$$k^n + 1 > k^n - k^m + k^{n-m} - 1 = (k^{n-m} - 1)(k^m + 1).$$

Thus $k^{n-m} - 1 < 10$ and, since $k \neq 10$, we have $k^{n-m} - 1 < 9$. But by the lemma the number

$$(k^n + 1) - (k^m + 1) = (k^{n-m} - 1)k^m$$

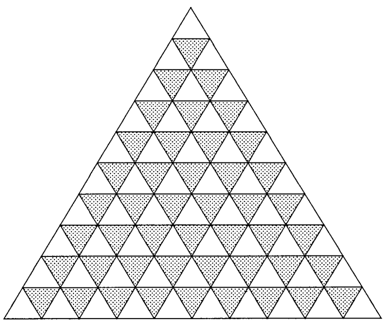
is a multiple of 9, hence k^m is divisible by 3. If $k \geq 6$, we get $n - m = 1$. Since

$$k^m + 1 < (k-1)(k^m + 1) < k^{m+1} + 1 = k^n + 1,$$

the number k^m+1, has the same number of digits as $(k-1)(k^m+1)$. Therefore its first digit must be 1. But this means that the last digit of $k^n + 1$ is 1, which is impossible since k is not a multiple of 10. It remains only to consider the only one possibility: $k = 3$. The numbers $m = 3$ and $n = 4$ satisfy the required conditions as $3^3 + 1 = 28$ and $3^4 + 1 = 82$.

21. Answer: (b) It is possible for odd m such that $5 \leq m \leq 25$ (and hence for (a) the answer is "no.")

Let us colour the small triangles in black and white as shown in the diagram:

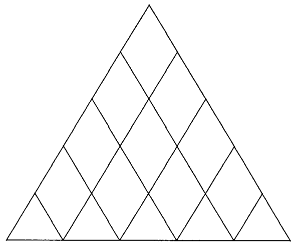


A tile of the second kind covers two white and two black cells. Let us denote by x the number of tiles of the first kind which cover 3 white cells. Then the total number of white cells is

$$3x + 1 \times (m - x) + 2 \times (25 - m) = 55,$$

or $2x = 5 + m$, hence m is odd. Furthermore $2m \geq 2x = 5 + m$, so $m \geq 5$.

If m is an odd number between 5 and 25, divide the original triangle into 5 tiles of the first kind and 10 rhombi of sidelength 2 (for instance, as shown in the following diagram).



Then it remains only to divide $(m - 5)/2$ of the rhombi into tiles of the first kind and the rest into tiles of the second kind.

22. Let the sum of all of the vectors be $\vec{a}$. Let us introduce a rectangular coordinate system, with the x-axis in the direction of $\vec{a}$ (or in any direction if $\vec{a} = \vec{0}$). At each move the first player can choose the vector with the largest x-coordinate. At the end of the game the sum of his vectors (the red vector) will have an x-coordinate not less than that of the sum of the second player's vectors (the green vector). Moreover, since the sum of all of the vectors is zero, the absolute values of the y-coordinates of each player's vectors are equal. Hence the first player cannot lose this game using this strategy.

23. Let $l = k + a$, $m = k + b$, $n = k + c$, for some positive integers a, b, c. Then $kn = lm$ implies

$$k(k + c) = (k + a)(k + b)$$

and $kc = k(a + b) + ab > k(a + b)$. Thus $c > a + b$ and hence $c - a - b \geq 1$. This implies that the inequality

$$(n + k) - (m + l) \geq 1$$

holds. We shall use this in the following proof of the required inequality.

Using $kn = lm$ again we get

$$(n+k)^2 - (m+l)^2 = (n-k)^2 - (m-l)^2$$

$$= ((n-k)+(m-l))\,((n-m)+(l-k)) > 0,$$

hence

$$(n-k)^2 = (n+k)^2 - 4ml > (n+k)^2 - (m+l)^2$$

$$= ((n+k)-(m+l))\,(k+l+m+n)$$

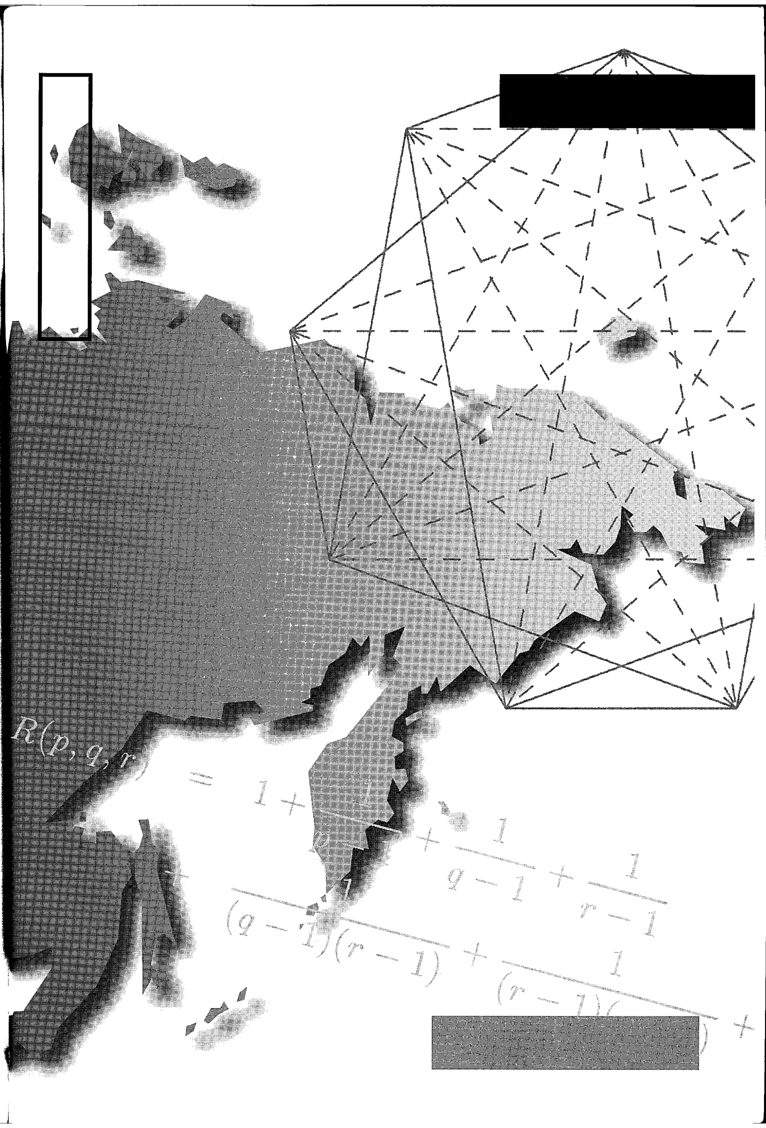
$$\geq 1 \times (k+(k+1)+(k+2)+(k+3)) = 4k+6.$$

But the square of an integer cannot have remainder 2 or 3 on division by 4. Therefore,

$$(n-k)^2 \geq 4k+8,$$

as was to be proved.

24. See the solution of Question 16.



IMO 1992

First day

Moscow, 15 July 1992

1. Find all integers a, b, c with $1 < a < b < c$ such that

$$(a-1)(b-1)(c-1)$$

is a divisor of $abc - 1$.

(New Zealand)

2. Let $\mathbb{R}$ denote the set of all real numbers. Find all functions

$$f : \mathbb{R} \mapsto \mathbb{R}$$

such that

$$f\left(x^2 + f(y)\right) = y + (f(x))^2$$

for all x, y in $\mathbb{R}$.

(India)

3. Consider 9 points in space, no 4 of which are coplanar. Each pair of points is joined by an edge (that is, a line segment), and each edge is either coloured blue or red or left uncoloured. Find the smallest value of n such that, whenever exactly n edges are coloured, the set of coloured edges necessarily contains a triangle all of whose edges have the same colour.

(China)

Time allowed: $4\frac{1}{2}$ hours
Each problem worth 7 points

Second day

Moscow, 16 July 1992

4. In the plane let c be a circle, l a line tangent to the circle c, and M a point on ℓ. Find the locus of all points P with the following property:

> There exist two points Q, R on ℓ such that M is the mid-point of QR and c is the inscribed circle of the triangle PQR.

(France)

5. Let S be a finite set of points in three-dimensional space. Let S_x, S_y, S_z be the sets consisting of the orthogonal projections of the points of S onto the yz-plane, zx-plane, xy-plane, respectively. Prove that

$$|S|^2 \leq |S_x| \cdot |S_y| \cdot |S_z|,$$

where $|A|$ denotes the number of elements in the finite set A.

(Note: the orthogonal projection of a point onto a plane is the foot of the perpendicular from that point to the plane).

(Italy)

6. For each positive integer n, $S(n)$ is defined to be the greatest integer such that, for every positive integer $k \leq S(n)$, n^2 can be written as the sum of k positive square integers.

(a) Prove that $S(n) \leq n^2 - 14$ for each $n \geq 4$.

(b) Find an integer n such that $S(n) = n^2 - 14$.

(c) Prove that there are infinitely many integers n such that $S(n) = n^2 - 14$.

(United Kingdom)

Time allowed: $4\frac{1}{2}$ hours
Each problem worth 7 points

SOLUTIONS

1. Let

$$R(p,q,r) = \frac{pqr - 1}{(p-1)(q-1)(r-1)}.$$

Suppose that $1 < p < q < r$, and that p, q, r and $R(p,q,r)$ are all integers. First, note that

$$\begin{aligned} R(p,q,r) &= 1 + \frac{1}{p-1} + \frac{1}{q-1} + \frac{1}{r-1} \\ &+ \frac{1}{(q-1)(r-1)} + \frac{1}{(r-1)(p-1)} + \frac{1}{(p-1)(q-1)}, \end{aligned}$$

from which it follows that $R(p,q,r) > 1$. It is also clear that $R(p,q,r) \leq R(p',q',r')$, if $p \geq p' \geq 1$, $q \geq q' \geq 1$, $r \geq r' \geq 1$. Secondly, note that $pqr - 1$ is odd unless p, q, r are all odd, and that $(p-1)(q-1)(r-1)$ is even unless p, q, r are all even. Since $R(p,q,r)$ is an integer, this implies that p, q, r are either all even or all odd.

If $p \geq 4$, then $1 < R(p,q,r) \leq R(4,6,8) = (4\times6\times8-1)/(3\times5\times7) = 191/105 < 2$, so $R(p,q,r)$ cannot be an integer.

If $p = 3$ then $1 < R(p,q,r) \leq R(3,5,7) = (3\times5\times7-1)/(2\times4\times6) = 104/48 < 3$, so $R(p,q,r) = R(3,q,r) = (3qr-1)/(2(q-1)(r-1)) = 2$, hence $3qr - 1 = 4(qr - q - r + 1)$, which can be written as $qr - 4q - 4r + 5 = 0$ or $(q-4)(r-4) = 11$. This implies that $q = 5$ and $r = 15$, since 11 is prime and $5 \leq q < r$.

If $p = 2$ then $1 < R(p,q,r) \leq R(2,4,6) = (2\times4\times6-1)/(1\times3\times5) = 47/15 < 4$, so $R(p,q,r) = R(2,q,r) = 2$ or 3. In the former case $2qr - 1 = 2(q-1)(r-1)$, which is impossible. In the latter case $2qr - 1 = 3(q-1)(r-1)$, which can be written as $qr - 3q - 3r + 4 = 0$ or $(q-3)(r-3) = 5$. It follows that $q = 4$ and $r = 8$ since 5 is prime and $4 \leq q < r$.

We conclude that there are at most two solutions, namely $(p,q,r) = (3,5,15)$ and $(p,q,r) = (2,4,8)$. It is easily verified that these are indeed solutions.

2. Let $s = f(0)$. Substituting $x = 0$ into the equation, we get

$$f(f(y)) = y + s^2 \quad \text{for all } y \text{ in } \mathbb{R}. \tag{1}$$

By setting $y = 0$ we also get

$$f(x^2 + s) = f(x)^2 \quad \text{for all } x \text{ in } \mathbb{R}, \tag{2}$$

and specifying further $x = 0$ we obtain

$$f(s) = s^2. \tag{3}$$

Let x be an arbitrary real number. Adding (2) and (3) gives

$$s^2 + f(x^2 + s) = f(x)^2 + f(s).$$

Now applying f to both sides and using (1), we get

$$(x^2 + s) + s^4 = s + f(f(x))^2 = s + (x + s^2)^2,$$

hence $2xs^2 = 0$ and, since x is arbitrary, $s = 0$, i.e.

$$f(0) = 0.$$

Therefore (1) and (2) can now be written as

$$f(f(y)) = y \quad \text{for all } y \text{ in } \mathbb{R}, \tag{4}$$

and

$$f(x^2) = f(x)^2 \quad \text{for all } x \text{ in } \mathbb{R}. \tag{5}$$

Equation (5) immediately implies that

$$f(x) \geq 0 \quad \text{for all } x \geq 0.$$

If $f(x) = 0$ for some x, then (3) and (4) immediately give

$$0 = f(0) = f(f(x)) = x.$$

Thus we conclude that

$$f(x) > 0 \quad \text{for all } x > 0. \tag{6}$$

Replacing y by $f(y)$ in the original functional equation and making use of (4) and (5), we get

$$f(x^2 + y) = f(y) + f(x^2),$$

or

$$f(x + y) = f(x) + f(y) \quad \text{for all } x \geq 0 \text{ and all } y \text{ in } \mathbb{R}. \tag{7}$$

Using this partial linearity we can prove that our function must be monotone and strictly increasing. Suppose that $x > y$. Then $x - y > 0$, and therefore (6) and (7) imply

$$f(x) = f((x - y) + y) = f(x - y) + f(y) > f(y).$$

Thus we have

$$f(x) > f(y) \quad \text{if} \quad x > y.$$

If $f(x) > x$, then $x = f(f(x)) > f(x)$. Similarly $f(x) < x$ leads to the conclusion that $x < f(x)$. We conclude that the only possibility is that

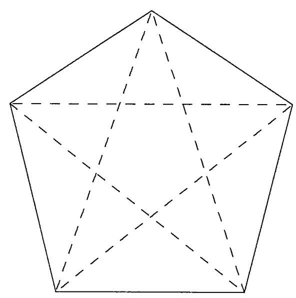
$$f(x) = x \quad \text{for all } x \text{ in } \mathbb{R}.$$

3. Any n points, all of which are connected by edges, will be called a complete graph on n vertices and will be denoted G_n. An arbitrary graph on n vertices can be obtained by removing some of the edges of G_n. It is known that if the edges of the complete graph G_6 are coloured with 2 colours, then there exist three edges, which form a monochrome triangle. If you are not familiar with this piece of olympiad folklore, it is time to stop reading and prove this as a warm-up exercise.

The complete graph G_9 on 9 vertices has $\frac{9 \times 8}{2} = 36$ edges. Suppose that 33 of them are coloured in blue or red. Then only three edges will be uncoloured, and we can choose three vertices which are endpoints of the first, the second and the third uncoloured edges. If we remove these three vertices together with the edges connecting them to the other points, we obtain a complete two-coloured graph on 6 vertices, which as we know contains a monochrome triangle.

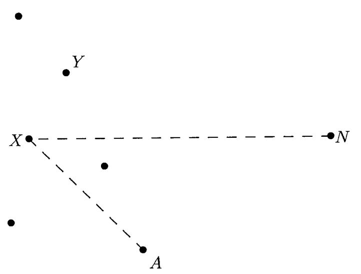
We will now prove that $n = 33$ is the smallest value with this property. We shall construct an example of a two-coloured graph on 9 vertices with 32 edges and no monochrome triangles.

We start with the complete graph G_5 on 5 vertices coloured as in the diagram:



where solid and dashed segments represent red and blue segments respectively. By inspection we see that it does not contain monochrome triangles. Now we shall describe a construction which, applied to a two-coloured graph H_n on n vertices with no monochrome triangles, but with a vertex A which is connected to every other vertex, gives a two-coloured graph H_{n+1} on $n+1$ vertices which also has no monochrome triangles.

We add a new vertex N to H_n and join it to all vertices of H except A. For an arbitrary vertex X of H we colour the new edge NX exactly as AX was coloured (see the diagram below).



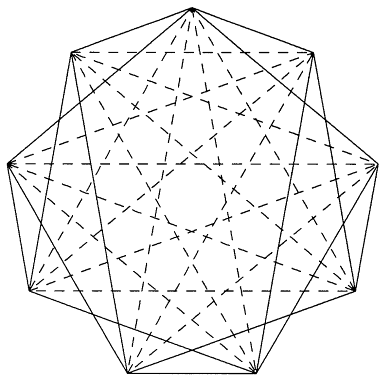
Note that no new monochrome triangles appear. For example, NXY cannot be monochrome because AXY was not.

We apply this construction four times starting from $H_5 = G_5$:

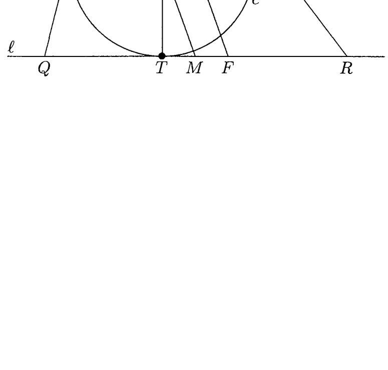
$$H_5 \longrightarrow H_6 \longrightarrow H_7 \longrightarrow H_8 \longrightarrow H_9.$$

The graph H_6 will be the complete graph G_6 with one edge deleted, H_7 will be G_7 with two edges deleted, H_8 will be G_8 with three edges deleted, and, finally, H_9 will be G_9 with four edges deleted. Note that at each step of this construction we can choose a vertex which is connected to every other vertex.

Thus H_9 is a two-coloured graph on 9 vertices with 32 edges and no monochrome triangles. It is shown below.



4. Let T be the point of tangency of the circle c to the line ℓ, E be the point on the circle which is opposite to T, and O be the centre of the circle. Let r be the ray originating at E in the direction of the vector $\overrightarrow{MO}$. We shall prove that the ray r is the locus of points sought.



Take an arbitrary point P on r, draw tangents to the circle c and let Q and R be the points of intersection of these tangents with the line ℓ (see the diagram). Without loss of generality we may assume that the point M is to the right of T, as the case $M = T$ is trivial. Let us also draw EF parallel to MO and the tangent to c at the point E which intersects PQ and PR at G and H, respectively. Since $SP = UP$, $SG = GE$ and $UH = HE$ we see that both $PG + GE$ and $PH + HE$ are equal to the semiperimeter of the triangle PGH. But the triangle PQR is homothetic to PGH and therefore $PQ + QF$ and $PR + RF$ are both equal to the semiperimeter p of PQR. Thus $RF = p - PR$. On the other hand $QT = \frac{1}{2}(QT + QS) = \frac{1}{2}(2p - (SP + PU + UR + RT) = p - PU - UR = p - PR$ and $RF = QT$. Since M is the midpoint of TF we have $QM = MR$.

If the point P does not lie on the ray r, then the point F will lie to the left or the right while the points T and M will be in the same places. Therefore in this case $TM \neq MF$ which implies $QM \neq MR$.

5. Both solutions that we present rely on Cauchy's inequality

$$(x_1y_1 + x_2y_2)^2 \leq \left(x_1^2 + x_2^2\right)\left(y_1^2 + y_2^2\right).$$

Solution 1. We shall prove the statement by induction on $|S|$. Let $|S_x| = a$, $|S_y| = b$, $|S_z| = c$. For a set S consisting of just one point, the statement is true. Suppose that the statement is true for all sets with fewer than n points. Consider a set S with $|S| = n$. Choose a plane parallel to one of the coordinate planes, which contains no points of S, and divides S into two non-empty subsets S_1 and S_2. Then $n = |S_1| + |S_2|$, where $|S_1| < n$ and $|S_2| < n$. By the induction hypothesis

$$|S_1|^2 < a_1b_1c_1, \qquad |S_2|^2 < a_2b_2c_2,$$

where a_i, b_i, c_i, are the number of elements in the projections of S_i, $i = 1, 2$, onto the coordinate planes yz, zx and xy, respectively. Without loss of generality we may assume that the dividing plane is parallel to the coordinate plane xy. Then

$$a_1 + a_2 = a, \quad b_1 + b_2 = b, \quad c_1 < c, \quad c_2 < c,$$

and by Cauchy's inequality

$$\begin{aligned} |S|^2 &= (|S_1| + |S_2|)^2 \leq \left(\sqrt{a_1b_1c_1} + \sqrt{a_2b_2c_2}\right)^2 \\ &\leq \left(\sqrt{a_1b_1}\sqrt{c} + \sqrt{a_2b_2}\sqrt{c}\right)^2 \leq c(a_1 + a_2)(b_1 + b_2) \\ &= abc. \end{aligned}$$

Solution 2. Let S_{ij} be the set of points in S of the type (x, i, j), i.e., the set of points whose orthogonal projection onto the yz-plane is the point (i, j). Clearly,

$$S = \bigcup_{(i,j)\in S_x} S_{ij},$$

and by Cauchy's inequality

$$\begin{aligned} |S|^2 &= \left(\sum_{(i,j)\in S_x} 1 \cdot |S_{ij}|\right)^2 \leq \sum_{(i,j)\in S_x} 1^2 \cdot \sum_{(i,j)\in S_x} |S_{ij}|^2 \\ &= |S_x| \cdot \sum_{(i,j)\in S_x} |S_{ij}|^2. \end{aligned}$$

Now we consider a set of cardinality $\sum_{(i,j)\in S_x} |S_{ij}|^2$, namely the set

$$X = \bigcup_{(i,j)\in S_x} (S_{ij} \times S_{ij}).$$

The map $f : X \mapsto S_y \times S_z$, given by

$$f((x, i, j), (x', i, j)) = ((x, j), (x', i))$$

is clearly injective, hence $|X| \leq |S_y| \cdot |S_z|$ and

$$|S|^2 \leq |S_x| \cdot |X| \leq |S_x| \cdot |S_y| \cdot |S_z|.$$

6. (a) It is enough to prove that n^2 can never be written as a sum of $n^2 - 13$ squares. Suppose on the contrary that $n^2 = a_1^2 + a_2^2 + \ldots + a_{n^2-13}^2$ for some positive integer a_i's. Then $1 \leq a_i \leq 3$ and we can write this equation as

$$n^2 = \underbrace{1^2 + \ldots + 1^2}_{p \text{ times}} + \underbrace{2^2 + \ldots + 2^2}_{q \text{ times}} + \underbrace{3^2 + \ldots + 3^2}_{r \text{ times}},$$

with $p + q + r = n^2 - 13$. Therefore the non-negative integers p, q, r satisfy

$$\begin{cases} p + q + r = n^2 - 13 \\ p + 4q + 9r = n^2. \end{cases}$$

This implies that $3q + 8r = 13$, which is impossible for non-negative integers q and r.

(b) We shall prove that $S(13) = 13^2 - 14 = 169 - 14 = 155$ by proving that 169 can be represented as a sum of k squares for $1 \le k \le 155$.

Suppose first that $43 \le k \le 155$. We shall look for a solution of the form

$$169 = \underbrace{1^2 + \ldots + 1^2}_{p \text{ times}} + \underbrace{2^2 + \ldots + 2^2}_{q \text{ times}} + \underbrace{3^2 + \ldots + 3^2}_{r \text{ times}},$$

where $p + q + r = k$. For such a representation to exist it is necessary and sufficient that the non-negative integers p, q, r satisfy

$$\begin{cases} p + q + r = k \\ p + 4q + 9r = 169. \end{cases}$$

Note that every integer greater than 13 can be written in the form $3x + 8y$, where x, y are non-negative integers (prove it by induction!). Since $169 - k \ge 14 > 13$, we can find numbers q, r such that $3q + 8r = 169 - k$. Then

$$4q + 9r = \frac{4}{3}(3q + \frac{27}{4}r) \le \frac{4}{3}(3q + 8r) \le \frac{4}{3}(169 - 43) = 168 < 169.$$

Therefore $p = 169 - 4q - 9r > 0$ and the conditions are satisfied since $p + q + r = 169 - 3q - 8r = 169 - (169 - k) = k$.

Suppose now that $19 \le k \le 42$. As in the previous case we shall look for a solution of the form

$$169 = \underbrace{1^2 + \ldots + 1^2}_{p \text{ times}} + \underbrace{3^2 + \ldots + 3^2}_{q \text{ times}} + \underbrace{4^2 + \ldots + 4^2}_{r \text{ times}}.$$

Again a necessary and sufficient condition for such a representation to exist is that the non-negative integers p, q, r satisfy

$$\begin{cases} p + q + r = k \\ p + 9q + 16r = 169. \end{cases}$$

Again by induction it can be proved that every positive integer, greater than 97, can be written in the form $8x + 15y$, where x, y are non-negative integers. Let $8q + 15r = 169 - k \ge 127 > 97$ be such a representation of the number $169 - k$. Then

$$\begin{aligned} 9q + 16r &= \frac{9}{8}\left(8q + \frac{128}{9}r\right) \\ &\le \frac{9}{8}(8q + 15r) \\ &\le \frac{9}{8}(169 - 19) < 169. \end{aligned}$$

Setting $p = 169 - (9q + 16r)$ we have $p > 0$, and the conditions are satisfied.

Finally suppose that $1 \le k \le 18$. We shall give the corresponding representations explicitly:

$$\begin{aligned} 169 &= 13^2 \\ &= 12^2 + 5^2 = 12^2 + 4^2 + 3^2 = 10^2 + 8^2 + 2^2 + 1^2 \\ &= 4 \times 6^2 + 5^2 = 4 \times 6^2 + 4^2 + 3^2 \\ &= 4 \times 5^2 + 8^2 + 2^2 + 1^2 = 4 \times 3^2 + 3 \times 6^2 + 5^2 \\ &= 4 \times 3^2 + 3 \times 6^2 + 4^2 + 3^2 \\ &= 4 \times 5^2 + 4 \times 4^2 + 2^2 + 1^2 = 8 \times 3^2 + 2 \times 6^2 + 5^2 \\ &= 8 \times 3^2 + 2 \times 6^2 + 4^2 + 3^2 \\ &= 4 \times 5^2 + 3 \times 4^2 + 5 \times 2^2 + 1^2 \\ &= 12 \times 3^2 + 6^2 + 5^2 = 12 \times 3^2 + 6^2 + 4^2 + 3^2 \\ &= 4 \times 5^2 + 2 \times 4^2 + 9 \times 2^2 + 1^2 = 16 \times 3^2 + 5^2 \\ &= 16 \times 3^2 + 4^2 + 3^2. \end{aligned}$$

The easiest way to check these representations is to check the first four representations directly and note that the $(k + 3)$th representation is obtained from the kth by substituting $n^2 + n^2 + n^2 + n^2$ for $(2n)^2$.

(c) **Solution 1.** We shall prove that if $n \ge 13$ and $S(n) = n^2 - 14$, then $S(2n) = (2n)^2 - 14$. This will provide us with an infinite series

$$169,\ 2 \times 169,\ \ldots,\ 2^k \times 169,\ \ldots$$

of numbers m such that $S(m) = m^2 - 14$.

We first note that since $(2n)^2 = n^2 + n^2 + n^2 + n^2$ and since each n^2 can be written as a sum of $1, 2, \ldots, n^2 - 14$ squares, then $(2n)^2$ can be written as a sum of $1, 2, \ldots, 4n^2 - 56$ squares, that is $S(2n) \ge (2n)^2 - 56$. Suppose now that $4n^2 - 56 < k \le 4n^2 - 14$. Since our n is large enough to secure the inequality $4n^2 - 56 > 3n^2$, it is sufficient to prove that $4n^2$ can be written as a sum of $3n^2 + k$ squares for $1 \le k \le n^2 - 14$. We represent $4n^2$ as follows:

$$\begin{aligned} 4n^2 &= \underbrace{1^2 + 1^2 + \ldots + 1^2}_{3n^2 \text{ times}} + n^2 \\ &= \underbrace{1^2 + 1^2 + \ldots + 1^2}_{3n^2 \text{ times}} + a_1^2 + \ldots + a_k^2, \end{aligned}$$

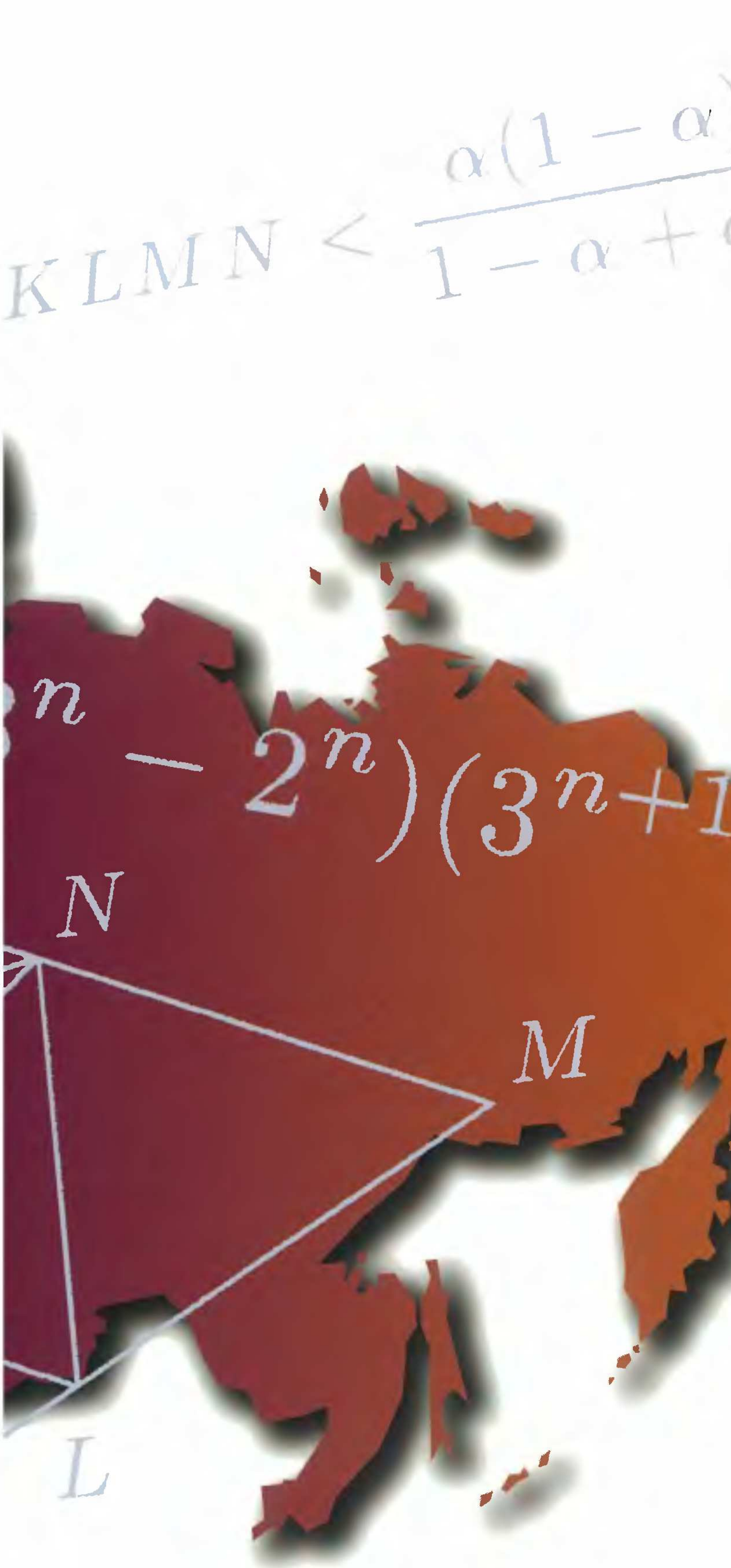
where $n^2 = a_1^2 + \ldots + a_k^2$ is a representation of n^2 as a sum of k squares.

Solution 2. This solution makes use of Lagrange's theorem that every positive integer can be represented as a sum of 4 squares of integers (some of which might be zeros). Some students knew this theorem and used it as, according to the rules of IMO, you can use any theorem you are familiar with.

First, we shall prove that any positive integer $n \geq 169$ can be written as a sum of 5 positive squares. For $n = 169$ this has already been proved in (b). If $n > 169$, using Lagrange's theorem, we represent $n - 169$ as a sum of $1, 2, 3$ or 4 positive integers. Then we add 169 represented as a sum of 4, 3, 2 or 1 squares of positive integers respectively.

Now, by induction, we can prove that every integer $n \geq 169$ can be represented as a sum of k positive squares for $5 \leq k \leq n - 14$. For $n = 169$ this has already been proved in (b). Suppose that $n \geq 169$ can be represented as a sum of k positive squares for $5 \leq k \leq n - 14$, then $n + 1 = n + 1^2$ can be represented as a sum of k positive squares for $6 \leq k \leq (n + 1) - 14$. But we have proved that $n + 1$ can also be represented as a sum of 5 positive squares, hence the induction is complete.

It is now sufficient to find an infinite sequence of integers greater than 169 which can be written as a sum of $1, 2, 3$ and 4 positive squares. We can take the numbers $169m^2$, $m = 1, 2, \ldots$.



The All-Union Olympiad of the USSR, before the breakup of the USSR was perhaps the most famous of all National Mathematical Olympiads. There was great competition among students even to compete, and the standard and range of questions was arguably higher than that found in the International Mathematical Olympiad.

Arkadii Slinko was highly involved in these last few years of the event and in this book he not only provides all the questions, but also presents highly expository solutions. This book is a must for teachers and all students aspiring to the highest Olympiad Challenges.

Arkadii Slinko graduated with PhD (1973) and DSc (1990) from the Institute of Mathematics in Novosibirsk. He has held a number of positions, having been Associate Professor at the Novosibirsk State University, Moscow Pedagogical University and Moscow State University. He currently has an academic appointment at the University of Auckland.

He has held a number of key positions on Olympiad Juries, including that of the USSR Mathematical Olympiad and was Coordinator-in-Chief of the International Mathematical Olympiad held in Moscow in 1992.

His research interests include Lie and Jordan algebras and coalgebras, contemporary algebraic structures, topological algebras, design of experiments and cluster analysis, and mathematics education of gifted students and mathematics competitions.

His publications include 42 books and refereed journal articles.

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